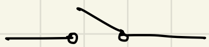
Transistors and Logic

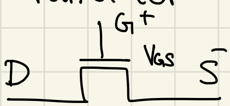
How do we implement computation?

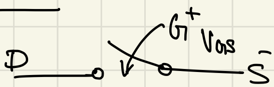
→ map numbers to distinct voltage levels (binary)

In 16A: Switch  → gives two models

"ON": wire (short circuit), "OFF": open circuit

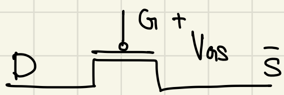
In 16B: Transistor

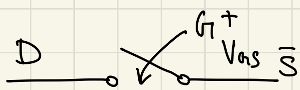
NMOS: 

simplest model: 

"ON" and "OFF" states

For some $V_{thn} \geq 0$, $V_{GS} \geq V_{thn} \rightarrow$ ON, $V_{GS} < V_{thn} \rightarrow$ OFF

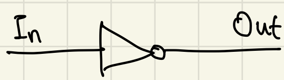
PMOS: 

→ 

For some $V_{thp} \leq 0$, $|V_{GS}| \geq |V_{thp}| \rightarrow$ ON, $|V_{GS}| < |V_{thp}| \rightarrow$ OFF

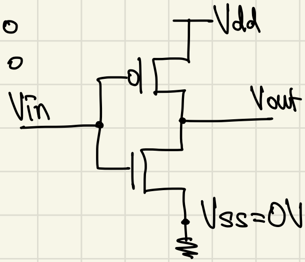
How do we use transistors to make digital logic?

Simplest logic operation: NOT

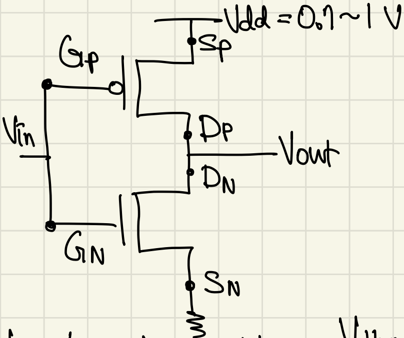
Simplest logic gate: Inverter 

$Out = \overline{In}$

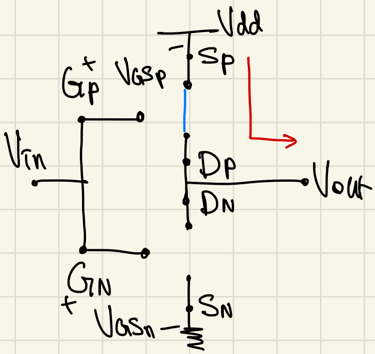
CMOS:



$\to$



(16B Assumption: $V_{thn} + |V_{thp}| \ge V_{dd}$, $V_{dd} \ge V_{thn}, |V_{thp}| \ge 0$)



1) $V_{in}=0$, logic 0 or False

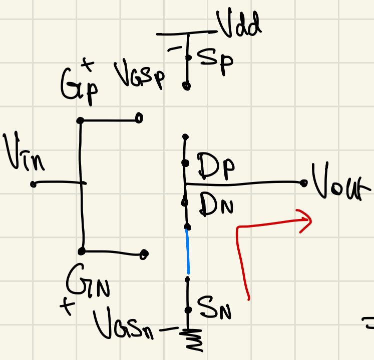
$V_{GSp} = V_{Gp} - V_{Sp} = \cancel{V_{in}}^{0} - V_{dd} = -V_{dd}$

$\to -V_{dd} \le -|V_{thp}| \to$ PMOS <u>ON</u>

$V_{GSn} = V_{Gn} - V_{Sn} = \cancel{V_{in}}^{0} - 0 = 0$

$\to 0 \le V_{thn} \to$ NMOS <u>OFF</u>

$\Rightarrow \underline{V_{out} = V_{dd}}$



2) $V_{in}=V_{dd}$, logic 1 or True

$V_{GSp} = V_{Gp} - V_{Sp} = 0 \to$ PMOS <u>OFF</u>

$V_{GSn} = V_{Gn} - V_{Sn} = V_{dd} \to$ NMOS <u>ON</u>

$\Rightarrow \underline{V_{out} = 0V}$

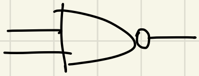
Truth Table:

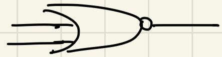
Vin	Vout
Vdd	0
0	Vdd

$\to$

In	Out
1	0
0	1

Other logic operations: NAND ($\overline{A \cdot B}$) NOR ($\overline{A+B}$)

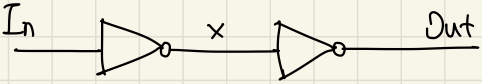




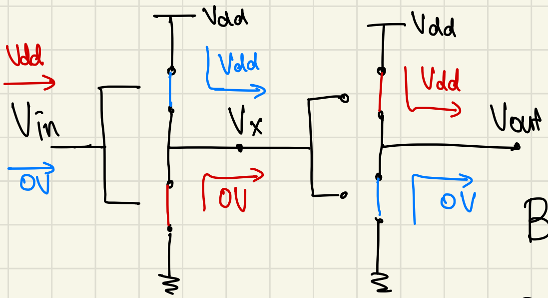
NAND and NOR are complete: can implement any function

Cascading Logic:

simplest model:



$x = \overline{In}$, $Out = \overline{x}$ → $Out = \overline{\overline{In}} = In$ (buffer)



1) $V_{in} = 0 \rightarrow V_x = V_{dd} \rightarrow V_{out} = 0$

2) $V_{in} = V_{dd} \rightarrow V_x = 0 \rightarrow V_{out} = V_{dd}$

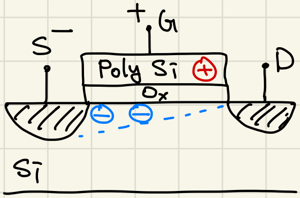
But... can you change voltages quickly enough so that  ?

In practice, output does not changes instantaneously! → not real

This model is good enough for logic functions, but not for figuring out speed & power of the device.

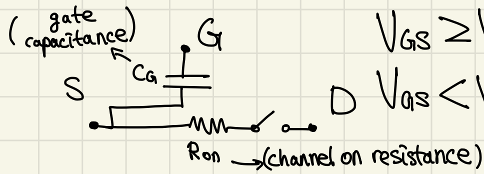
MOSFET: Metal-Oxide Semiconductor Field Effect Transistor

NMOS $\rightarrow$ n-channel MOSFET, PMOS $\rightarrow$ p-channel MOSFET



(NMOS)

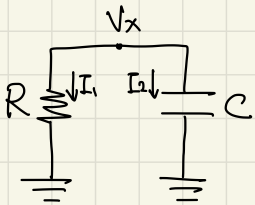
$\longrightarrow$



$V_{GS} \geq V_{th} \rightarrow \text{ON}$

$V_{GS} < V_{th} \rightarrow \text{OFF}$

RC Circuits



$$I_2 = C\frac{dV_x}{dt},\quad I_1 = \frac{V_x}{R},\quad I_1 + I_2 = 0$$

$$\rightarrow C\frac{dV_x}{dt} + \frac{V_x}{R} = 0 \rightarrow \underbrace{\frac{dV_x}{dt} = -\frac{1}{RC}V_x}$$

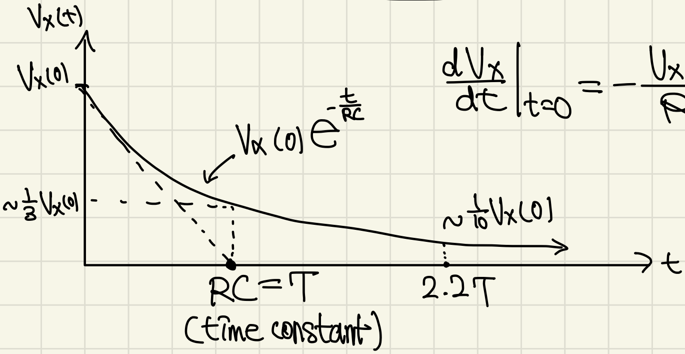
(first order differential equation!)

① Guess: $V_x(t) = ae^{bt}$, educated guess based on properties of e^x.

Initial Condition? $V_x(0) = ae^{b\cdot 0} = a$

$$\frac{d}{dt}V_x(t) = abe^{bt} = -\frac{1}{RC}V_x = -\frac{a}{RC}e^{bt} \rightarrow b = -\frac{1}{RC}$$

$$\Rightarrow \boxed{V_x(t) = V_x(0)\cdot e^{-\frac{t}{RC}}} \rightarrow \text{a solution to } \frac{dV_x}{dt} = -\frac{1}{RC}V_x$$



$$\left.\frac{dV_x}{dt}\right|_{t=0} = -\frac{V_x(0)}{RC} \rightarrow \text{tangent line at } t=0$$

has y-intercept $V_x(0)$,

x-intercept RC.

② Check for uniqueness of the guess:

Suppose $y(t)$ which also satisfies the diff. eq.

$\left(\overset{(1)}{X(0) = X_0}, \ \overset{(2)}{\frac{d}{dt}X(t) = \lambda X(t)}, \ V_X(t) = X(t), \ \lambda = -\frac{1}{RC} \right)$

In ① $X_d(t) = X_0 e^{\lambda t}$ satisfies the diff. eq.

→ Prove that $y(t) = X_d(t)$, i.e. the solution is unique

→ Eithe prove $\frac{y(t)}{X_d(t)} = 1$ or $y(t) - X_d(t) = 0$.

$\rightarrow \frac{y(t)}{X_d(t)} = \frac{y(t)}{X_0 e^{\lambda t}} \rightarrow \frac{d}{dt}\left(\frac{y(t)}{X_0 e^{\lambda t}}\right) = \frac{1}{X_0}\frac{d}{dt}\left(y(t) \cdot e^{-\lambda t}\right)$

$= \frac{1}{X_0}\left(\frac{dy}{dt} \cdot e^{-\lambda t} + y(t)(-\lambda)e^{-\lambda t}\right)$

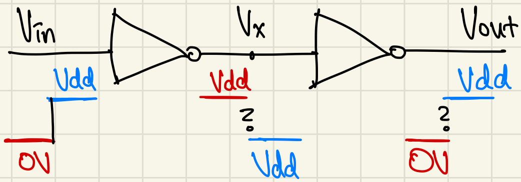
→ Since $y(t)$ is a solution, $\frac{d}{dt}y(t) = \lambda y(t)$ (2)

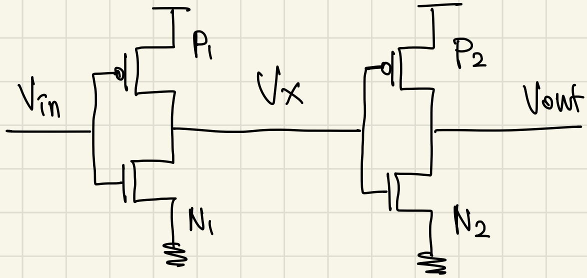
$\rightarrow \frac{1}{X_0}\left(\lambda y(t) \cdot e^{-\lambda t} - \lambda y(t) \cdot e^{-\lambda t}\right) = 0.$

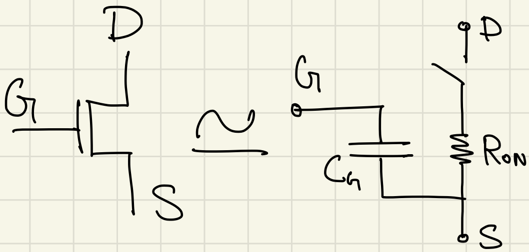
$\rightarrow \frac{y(t)}{X_0 e^{\lambda t}} = C$. using (1), $X(0) = X_0 = y(0)$, $\frac{y(0)}{X_d(0)} = \frac{X_0}{X_0} = 1.$

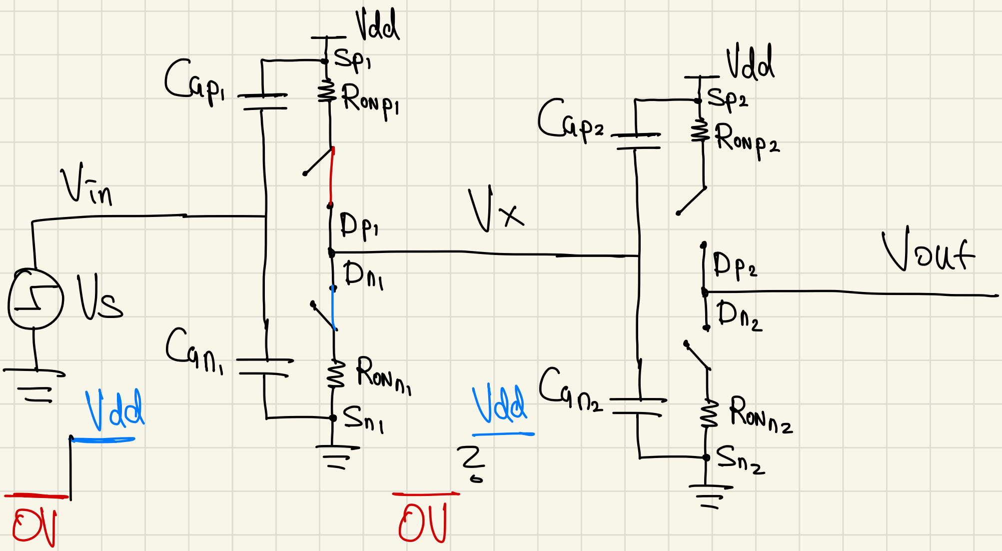
$\rightarrow \frac{y(t)}{X_d(t)} = 1 \Rightarrow y(t) = X_d(t)$ //

Now, use this to solve transistor models with RC circuits!

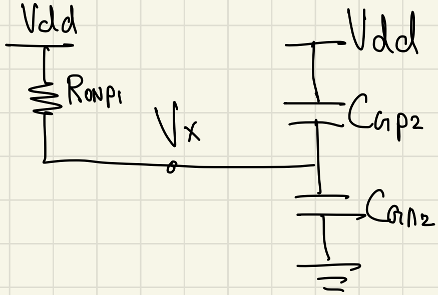

Vin
Vdd
0V
Vx
Vdd
?
Vdd
Vout
Vdd
?
0V


Vin
P1
N1
Vx
P2
N2
Vout


D
G
S
~
G
D
CG
RON
S

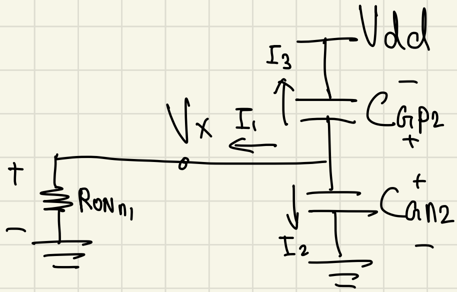

Vdd
Sp1
Cap1
Ronp1
Vin
Dp1
Dn1
Vs
Can1
Ronn1
Sn1
Vdd
0V
Vx
Vdd
?
0V
Vdd
Sp2
Cap2
Ronp2
Dp2
Vout
Dn2
Can2
Ronn2
Sn2

$V_s = 0$:
$(t<0)$



$V_x(0) = V_{dd}$

$V_s = V_{dd}$:
$(t \geq 0)$



$I_1 + I_2 + I_3 = 0$

$I_1 = \frac{V_x}{R_{ONn1}}$

$I_2 = C_{Gn2} \cdot \frac{d(V_x - 0)}{dt}$

$I_3 = C_{Gp2} \cdot \frac{d(V_x - V_{dd})}{dt}$

$\rightarrow \frac{V_x}{R_{ONn1}} + C_{Gn2}\frac{dV_x}{dt} + C_{Gp2}\frac{d}{dt}(V_x - \cancel{V_{dd}}) = 0$

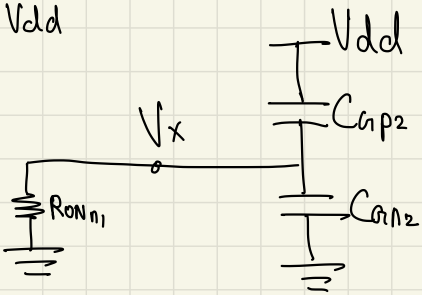
$\rightarrow \frac{V_x}{R_{ONn1}} + (C_{Gn2} + C_{Gp2})\frac{dV_x}{dt} = 0 \rightarrow \boxed{\frac{dV_x}{dt} = \frac{V_x}{-R_{ONn1}(C_{Gn2}+C_{Gp2})}}$

$\rightarrow \underline{\tau = R_{ONn1} \cdot (C_{Gn2} + C_{Gp2})}$

$\Rightarrow V_x(t) = \boxed{V_{dd}\, e^{-\frac{t}{\tau}}}$

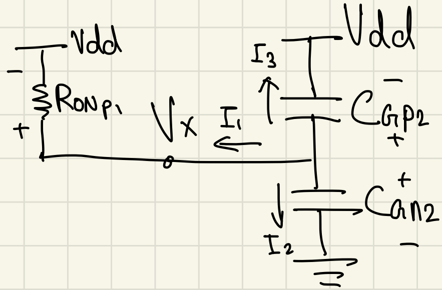
What about $V_s = V_{dd} \rightarrow V_s = 0V$?

$V_s = 0$:
$(t<0)$



$V_x(0) = 0V$

$V_s = V_{dd}$:
$(t \geq 0)$



$I_1 + I_2 + I_3 = 0$

$I_1 = \frac{V_x - V_{dd}}{R_{onp1}}$

$I_2 = C_{Gn2} \cdot \frac{d(V_x - 0)}{dt}$

$I_3 = C_{Gp2} \cdot \frac{d(V_x - V_{dd})}{dt}$

$$\rightarrow \frac{V_x}{R_{ONp1}} - \frac{V_{dd}}{R_{ONp1}} + C_{Gn2}\frac{d}{dt}V_x + C_{Gp2}\left(\frac{d}{dt}V_x - \cancel{\frac{d}{dt}V_{dd}}\right) = 0$$

$$\rightarrow \frac{V_x}{R_{ONp1}} + (C_{Gn2} + C_{Gp2})\frac{dV_x}{dt} = \frac{V_{dd}}{R_{ONp1}} \rightarrow \boxed{\frac{dV_x}{dt} = -\frac{V_x}{\Sigma C \cdot R} + \frac{V_{dd}}{\Sigma C \cdot R}}$$

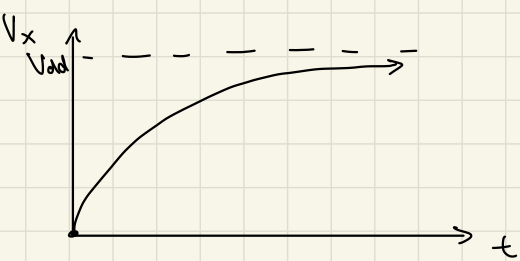
This is a nonhomogeneous diff. eq. of form $x' = \lambda x + a$

Go back to $\frac{V_x - V_{dd}}{R_{ONp1}} + (C_{Gn2} + C_{Gp2})\frac{d}{dt}(V_x - V_{dd}) = 0$

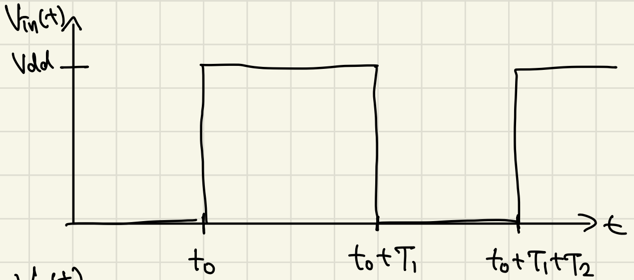
Set $\tilde{V}_x = V_x - V_{dd} \rightarrow \frac{\tilde{V}_x}{R_{ONp1}} + (C_{Gn2} + C_{Gp2})\frac{d}{dt}\tilde{V}_x = 0$

$$\rightarrow \tilde{V}_x = -\tilde{V}_x(0) \cdot e^{-\frac{t}{\tau}} \quad (\tau = R_{ONp1}(C_{Gn2} + C_{Gp2}))$$

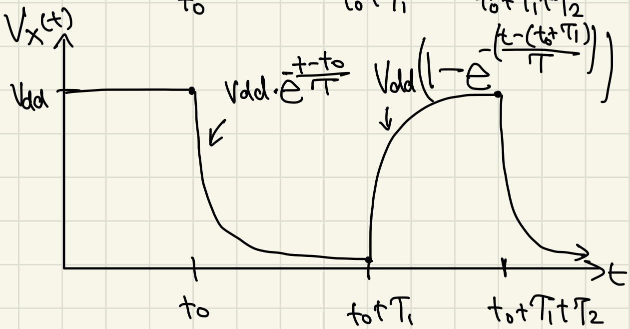
$$\Rightarrow V_x = V_{dd} - V_{dd}e^{-\frac{t}{\tau}} = V_{dd}(1 - e^{-\frac{t}{\tau}})$$



Now... how fast can signals change for transients to act as logic gates properly?



Will $V(x)$ be able to follow these changes as a logic signal?



Seems to work... at every T_i, V_x is nearly V_{dd} or 0.

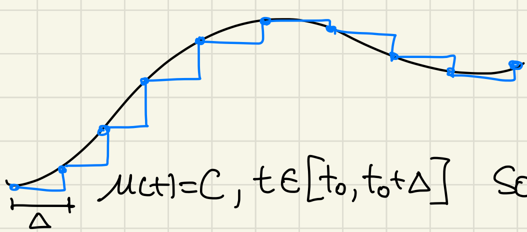
If T is too big to assume $V_x=0$ or V_{dd} for every T_i, then use previous $V_x(t)$ as initial condition for next.

Solution to piecewise constant output: $x' = \lambda x - \lambda u(t)$

What if $u(t) = u_c(t)$ where $u_c(t)$ is continuous?

$\rightarrow$ Approx. $u_c(t)$ into piecewise constant $u(t)$!

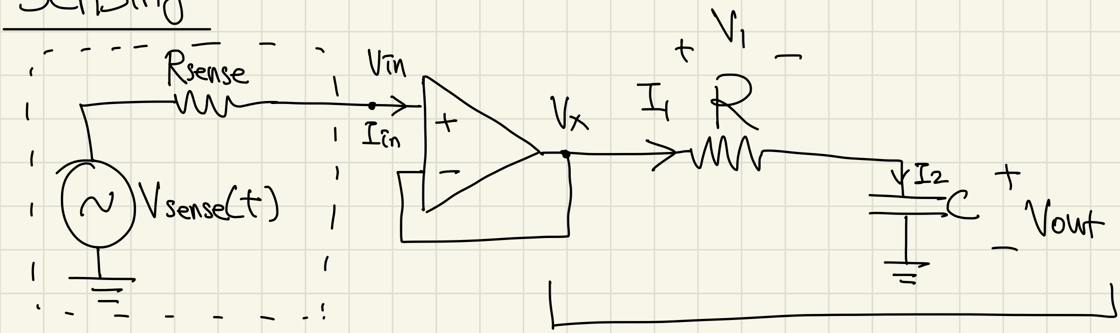
$$\lim_{\Delta \to 0} u(t) = u_c(t)$$



$u(t)=C,\ t\in[t_0, t_0+\Delta]$

Iterate & use previous solution, let $\Delta \to 0$ and observe.

Sensing



Want this circuit to separate low-frequency from high-frequency noise

$V_{in} = V_{sense} \; (\because I_{in} = 0)$

$V_x = V_{in}$ (buffer)

$I_1 = I_2$ (KCL), $V_1 = I_1 \cdot R$, $I_2 = \frac{dV_{out}}{dt} \cdot C$

$\rightarrow \frac{V_1}{R} = C\frac{dV_{out}}{dt} = \frac{V_x - V_{out}}{R} = C\frac{dV_{out}}{dt}$

$\rightarrow \frac{V_{sense} - V_{out}}{R} = C\frac{dV_{out}}{dt} \rightarrow \frac{dV_{out}}{dt} = -\frac{V_{out}}{RC} + \frac{V_{sense}}{RC}$

($V_{sense}(t)$ is a continuous time signal)

$$\Rightarrow V_{out}(t) = \underbrace{V_{out}(0)e^{-\frac{t}{RC}}}_{\substack{\text{homogeneous sol.} \\ \text{(response to initial condition)}}} + \underbrace{\frac{1}{RC}\int_0^t V_{sense}(t)e^{-\frac{1}{RC}(t-\theta)}d\theta}_{\substack{\text{nonhomogeneous sol.} \\ \text{(response to time input)}}}$$

The circuit "computes" this formula!

General Form: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda \mu(t)$

$\left(\lambda = -\frac{1}{RC},\ \mu(t) = V_{sense}(t),\ x(t) = V_{out}(t)\right)$

ex1) $\mu(t) = e^{st}\ (s \neq 0,\ t \geq 0)$

$\rightarrow x(t) = x(0)e^{\lambda t} - \lambda \int_0^t \mu(\theta) e^{\lambda(t-\theta)} d\theta$

$= x(0)e^{\lambda t} - \lambda e^{\lambda t} \int_0^t e^{s\theta} \cdot e^{-\lambda\theta} d\theta$

Guess and Check: $x(t) = Ke^{st},\ t \geq 0$

$\rightarrow K \cdot s e^{st} = \lambda \cdot K e^{st} - \lambda e^{st} \rightarrow Ks = K\lambda - \lambda$

$\rightarrow K = -\frac{\lambda}{s-\lambda} \Rightarrow x_\lambda(t) = -\frac{\lambda}{s-\lambda} e^{st}$

To complete, add a homogeneous solution:

$x(t) = K_2 e^{\lambda t} + K e^{st} \rightarrow x(0) = K_2 + K$

$\rightarrow K_2 = x(0) + \frac{\lambda}{s-\lambda}$

$\Rightarrow x(t) = \boxed{\left(x(0) + \frac{\lambda}{s-\lambda}\right)e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}}$

ex2) $\mu(t) = \cos(\omega t)\quad (t \geq 0)$

$\rightarrow x(t) = x(0)e^{\lambda t} - \lambda \int_0^t \cos(\omega\theta) e^{\lambda(t-\theta)} d\theta$

$= x(0)e^{\lambda t} - \lambda e^{\lambda t} \int_0^t \cos(\omega\theta) e^{-\lambda\theta} d\theta$

$\left[\int \cos(bx) e^{ax} dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin(bx) + a\cos(bx)\right)\right]$

$$x(t) = x(0)e^{\lambda t} - \lambda e^{\lambda t}\left\{ \frac{e^{-\lambda t}}{\lambda^2+w^2}\left(w\sin(wt) - \lambda\cos(wt)\right) - \frac{1}{\lambda^2+w^2}(0-\lambda) \right\}$$

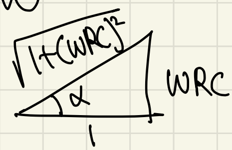
$$= x(0)e^{\lambda t} - \frac{\lambda}{\lambda^2+w^2}\left(w\sin(wt) - \lambda\cos(wt)\right) - \frac{\lambda^2}{\lambda^2+w^2}e^{\lambda t}$$

$$= \underbrace{\left(x(0) - \frac{\lambda^2}{\lambda^2+w^2}\right)e^{\lambda t}}_{①} \underbrace{- \frac{\lambda}{\lambda^2+w^2}\left(w\sin(wt) - \lambda\cos(wt)\right)}_{②}$$

As $t \to \infty$, ① $\to 0$ $(\lambda < 0)$

② for $\lambda = \frac{-1}{RC}$, $\dfrac{\frac{1}{RC}w\sin(wt) + \left(\frac{1}{RC}\right)^2\cos(wt)}{\left(\frac{1}{RC}\right)^2 + w^2}$

$$= \frac{wRC\sin(wt) + \cos(wt)}{1+(wRC)^2}$$



$$x(t) = \frac{1}{\sqrt{1+(wRC)^2}}\left(\frac{1}{\sqrt{1+(wRC)^2}}\cos(wt) + \frac{wRC}{\sqrt{1+(wRC)^2}}\sin(wt)\right)$$

$$= \frac{1}{\sqrt{1+(wRC)^2}}\left(\cos(\alpha)\cos(wt) + \sin(\alpha)\sin(wt)\right)$$

$$= \frac{1}{\sqrt{1+(wRC)^2}}\cos(wt-\alpha) = \boxed{\frac{1}{\sqrt{1+(wRC)^2}}\cos(wt+\theta)}$$

$(\theta = -\alpha = -\tan^{-1}(wRC))$

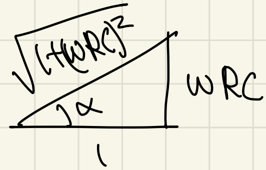
Case1) $w \gg \frac{1}{RC}$ $(wRC \gg 1) \rightarrow x(t) \simeq 0$

Case2) $w \ll \frac{1}{RC}$ $(wRC \ll 1) \rightarrow x(t) \simeq \cos(wt+\theta)$

$\Rightarrow$ This system is a "low pass" filter with $\frac{1}{RC}$ cutoff frequency!

Can also Guess and Check: $x(t) = A\cos(\omega t + \theta)$ for

$u(t) = V_{sense}\cos(\omega t)$



System: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t)$

$\rightarrow -A\omega\sin(\omega t+\theta) = \lambda A\cos(\omega t+\theta) - \lambda V_{sense}\cos(\omega t)$

$\rightarrow V_{sense}\cos(\omega t) = A\left(\cos(\omega t+\theta) + \frac{\omega}{\lambda}\sin(\omega t+\theta)\right)$

$\lambda = -\frac{1}{RC} \rightarrow V_{sense}\cos(\omega t) = A\left(\cos(\omega t+\theta) - \omega RC\sin(\omega t+\theta)\right)$

$\rightarrow V_{sense}\cos(\omega t) = A\sqrt{1+(\omega RC)^2}\left(\frac{1}{\sqrt{1+(\omega RC)^2}}\cos(\omega t+\theta) - \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}\sin(\omega t+\theta)\right)$

$\rightarrow V_{sense}\cos(\omega t) = A\sqrt{1+(\omega RC)^2}\left(\cos(\alpha)\cos(\omega t+\theta) - \sin(\alpha)\sin(\omega t+\theta)\right)$

$$= A\sqrt{1+(\omega RC)^2}\cos(\omega t + \theta + \alpha)$$

$\rightarrow V_{sense} = A\sqrt{1+(\omega RC)^2}, \ \omega t = \omega t + \theta + \alpha$

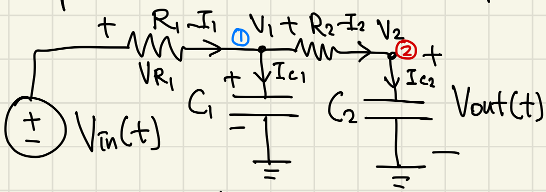
$\rightarrow A = \frac{V_{sense}}{\sqrt{1+(\omega RC)^2}}, \ \theta = -\alpha = -\tan^{-1}(\omega RC)$

$$\Rightarrow \boxed{x(t) = \frac{V_{sense}}{\sqrt{1+(\omega RC)^2}}\cos(\omega t + \theta)}$$

more general: $x(t) = x(t_0)e^{\lambda(t-t_0)} - \lambda\int_{t_0}^{t} u(\theta)e^{\lambda(t-\theta)}d\theta \ (t \geq t_0)$

Systems of Diff. Eqs

Example of Circuit: Two capacitor circuit.



KCL: $I_2 = I_{C_2}$, $I_1 = I_{C_1} + I_2$

Elements: $I_{C_1} = C_1 \frac{dV_1}{dt}$, $I_{C_2} = C_2 \frac{dV_2}{dt}$

① $\frac{V_{in} - V_1}{R_1} = C_1 \frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2}$ ② $\frac{V_1 - V_2}{R_2} = C_2 \frac{dV_2}{dt} \rightarrow V_1 = V_2 + R_2 C_2 \frac{dV_2}{dt}$

$\rightarrow$ 2^{nd} order diff. eq with $\frac{d^2 V_2}{dt}$! Let's go back...

$$\begin{cases} \frac{V_{in} - V_1}{R_1} = C_1 \frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2} \rightarrow \frac{dV_1}{dt} = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) V_1 + \frac{V_2}{R_2 C_1} + \frac{V_{in}}{R_1 C_1} \\ V_1 = V_2 + R_2 C_2 \frac{dV_2}{dt} \rightarrow \frac{dV_2}{dt} = \frac{V_1}{R_2 C_2} - \frac{V_2}{R_2 C_2} \end{cases}$$

$\rightarrow$ Write this in matrix form, $\frac{d}{dt}$ as an operator.

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \left(\frac{1}{R_2 C_1}\right) \\ \left(\frac{1}{R_2 C_2}\right) & -\left(\frac{1}{R_2 C_2}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C_1} \\ 0 \end{bmatrix} V_{in} \longrightarrow \dots ?$$

ex) Assume: $R_1 = \frac{1}{3} M\Omega$, $R_2 = \frac{1}{2} M\Omega$, $C_1 = C_2 = 1 \mu F$, $V_{in} = 1V \ (t<0)$, $V_{in} = 0V \ (t \geq 0)$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}, \quad V_1(0) = V_2(0) = 1V$$

What if we had a "magic" change such that...

$$u_1 = V_2, \ u_2 = V_1 + 2V_2 \rightarrow \underline{\frac{d}{dt} u_1} = \frac{d}{dt} V_2 = 2V_1 - 2V_2 = 2(u_2 - 2V_2) - 2V_2$$

$$= 2u_2 - 4V_2 - 2V_2 = 2u_2 - 6V_2 = \underline{-6u_1 + 2u_2}$$

$$\underline{\frac{d}{dt} u_2} = \frac{dV_1}{dt} + 2\frac{dV_2}{dt} = -5V_1 + 2V_2 + 4V_1 - 4V_2 = -V_1 - 2V_2 = \underline{-u_2} \rightarrow !$$

$$\rightarrow u_2(t) = u_2(0) e^{-t} \ (t \geq 0) \rightarrow u_2(0) = V_1(0) + 2V_2(0) = 3 \rightarrow \underline{u_2(t) = 3e^{-t}}$$

Use $u_2(t)$ to solve $\frac{d}{dt}u_1 = -6u_1 + 2u_2 \rightarrow \underline{\frac{d}{dt}u_1 = -6u_1 + 2(3e^{-t})} \rightarrow !$

$\rightarrow$ recall: $\frac{d}{dt}x(t) = \lambda x(t) - \lambda u(t),\ u(t) = e^{st} \rightarrow x(t) = k_2 e^{\lambda t} - \frac{\lambda}{s-\lambda} e^{st}$

$\rightarrow \lambda = -6,\ s = -1 \longrightarrow u_1(t) = k_2 e^{-6t} + \frac{6}{5}e^{-t},\ u_1(0) = k_2 + \frac{6}{5} = 1\,V$

$\rightarrow k_2 = -\frac{1}{5} \Rightarrow \underline{u_1(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}}$

Now, back-solve for $V_1(t)$ and $V_2(t)$ using $u_1(t)$ and $u_2(t)$.

$(u_1 = V_2,\ u_2 = V_1 + 2V_2) \rightarrow (V_1 = -2u_1 + u_2,\ V_2 = u_1)$

$\Rightarrow V_1(t) = 3e^{-t} - 2\left(-\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}\right) = \boxed{\frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t}}$

$V_2(t) = \boxed{-\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}}$

What is happening in matrix form?

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in},\quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\rightarrow \frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{d}{dt}\left(W^{-1}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right) = W^{-1}\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}}\left(\underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in}\right)$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}}_{W} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}}_{W^{-1}} \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\vec{b}} V_{in}$$

$$= \underbrace{\begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix}}_{W^{-1}AW} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{W^{-1}\vec{b}} V_{in} \rightarrow W^{-1}AW \text{ is upper-triangular!}$$

solve bottom-up $(u_2 \rightarrow u_1)$

In general: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$

"Nice" $\vec{\tilde{x}}$ coordinates, $\vec{x} = V\vec{\tilde{x}}$

$\underline{\frac{d}{dt}\vec{\tilde{x}} = V^{-1}AV\vec{\tilde{x}} + V^{-1}B\vec{u}(t)}$ ☆

```tikzcd
\vec{x} \arrow[r, "A"] \arrow[d, shift right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, shift right, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[r, "(V^{-1}AV)"] \arrow[u, shift right, "V"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, shift right, "V"']
\end{tikzcd}
```

How do we get $V^{-1}AV$ to be "nice"? (diagonal entries)
And how do the initial conditions transform?

$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) \rightarrow$ Apply this, then apply $\vec{x}(t) = V\vec{\tilde{x}}(t)$.

Diagonalization

Consider: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{B}u(t)$, $\vec{x}(0)$, $\vec{x}(t)$ for $t \geq 0$?

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, shift right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, shift right, "V^{-1}"'] \\
\vec{\tilde{x}} \arrow[u, shift right, "V"'] \arrow[r, "V^{-1}AV"'] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, shift right, "V"']
\end{tikzcd}$$

$$\vec{x} = V\vec{\tilde{x}} = \left[\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_n\right]\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \tilde{x}_1\vec{V}_1 + \cdots + \tilde{x}_n\vec{V}_n$$

$\rightarrow \vec{\tilde{x}}$ are coordinates in basis V.

$$\vec{\tilde{x}} = V^{-1}\vec{x} \rightarrow \frac{d}{dt}(\vec{\tilde{x}}) = \frac{d}{dt}(V^{-1}\vec{x}) = V^{-1}\frac{d}{dt}\vec{x}$$

$$= V^{-1}(A\vec{x} + B\vec{u}) = \underline{V^{-1}AV\vec{\tilde{x}}} + V^{-1}B\vec{u} \quad \text{(step 1)}$$

(step 2) $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0)$. $\Rightarrow$ Solve for $\vec{\tilde{x}}(t)$ when $V^{-1}AV$ is <u>diagonal</u> or <u>upper-triangle</u>.

(step 3) $\vec{x}(t) = V\vec{\tilde{x}}(t)$. //

- We want $V^{-1}AV$ to be diagonal (seperable equations!)

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

So how do we choose V?

$V^{-1}AV = V^{-1}A[\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n] = V^{-1}[A\vec{v}_1, A\vec{v}_2, \cdots, A\vec{v}_n]$

(Recall: if $A\vec{x} = \lambda\vec{x}$, λ is an eigen value of an eigenvector $\vec{x}$ of A.)

$$\rightarrow V^{-1}[\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \cdots, \lambda_n\vec{v}_n] = V^{-1}\underbrace{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]}_{V}\underbrace{\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}}_{\Lambda}$$

$= V^{-1}V\Lambda = \Lambda \Rightarrow \Lambda$ is diagonal!

(if $\vec{v}_i$s are independent eigenvectors of A)

So, if V is an eigenbasis of A, then $\frac{d}{dt}\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} + V^{-1}Bu$!

$\Longrightarrow$ This is a set of seperable equations!

ex) $A = \begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$ ① find eigenvalues of A. ($\det(A-\lambda I)=0$: $(a-\lambda)(d-\lambda)-bc=0$)

$\rightarrow (-5-\lambda)(-2-\lambda)-4=0 \rightarrow \lambda^2+7\lambda+6=0 \rightarrow \lambda=-1$ or -6

$\Rightarrow \Lambda = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \boxed{\begin{bmatrix}-1 & 0\\ 0 & -6\end{bmatrix}}$ ② find eigenvectors of λ_1 and λ_2.

$\lambda_1=-1$: $\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix} \rightarrow \vec{x}_2 = t\begin{bmatrix}1\\ 2\end{bmatrix}$, $\lambda_2=-6$: $\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \rightarrow \vec{x}_1 = t\begin{bmatrix}-2\\ 1\end{bmatrix}$

$\Rightarrow V = [\vec{v}_1\ \vec{v}_2] = \boxed{\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}}$ $\left(V^{-1} = \begin{bmatrix}1/5 & 2/5\\ -2/5 & 1/5\end{bmatrix}\right)$ ③ Solve $\vec{\tilde{x}}(t)$.

$\rightarrow \frac{d}{dt}\vec{\tilde{x}} = \Lambda\vec{\tilde{x}} = \begin{bmatrix}-1 & 0\\ 0 & -6\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} = \begin{bmatrix}-\tilde{x}_1\\ -6\tilde{x}_2\end{bmatrix} \rightarrow \begin{bmatrix}\frac{d}{dt}\tilde{x}_1\\ \frac{d}{dt}\tilde{x}_2\end{bmatrix} = \begin{bmatrix}-\tilde{x}_1\\ -6\tilde{x}_2\end{bmatrix} \rightarrow$ seperable!

$$\rightarrow \begin{cases}\frac{d}{dt}\tilde{x}_1 = -\tilde{x}_1\\ \frac{d}{dt}\tilde{x}_2 = -6\tilde{x}_2\end{cases} \rightarrow \begin{cases}\tilde{x}_1 = k_1e^{-t}\\ \tilde{x}_2 = k_2e^{-6t}\end{cases} \rightarrow \begin{cases}\tilde{x}_1 = \tilde{x}_1(0)e^{-t}\\ \tilde{x}_2 = \tilde{x}_2(0)e^{-6t}\end{cases} \rightarrow \vec{\tilde{x}}(t) = \begin{bmatrix}\tilde{x}_1(0)e^{-t}\\ \tilde{x}_2(0)e^{-t}\end{bmatrix}$$

* find $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix} \to \begin{cases} \tilde{x}_1(0) = 3/5 \\ \tilde{x}_2(0) = -1/5 \end{cases}$

$\to \vec{\tilde{x}}(t) = \boxed{\begin{bmatrix} 3/5\, e^{-t} \\ -1/5\, e^{-6t} \end{bmatrix}}$ ④ Convert back to $\vec{x}(t)$. ($\vec{x}(t) = V\vec{\tilde{x}}(t)$)

$$\vec{x}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\vec{\tilde{x}}(t) = \begin{bmatrix} 3/5\, e^{-t} + 2/5\, e^{-6t} \\ 6/5\, e^{-t} - 1/5\, e^{-6t} \end{bmatrix} \to \begin{cases} x_1(t) = 3/5\, e^{-t} + 2/5\, e^{-6t} \\ x_2(t) = 6/5\, e^{-t} - 1/5\, e^{-6t} \end{cases} \quad //$$

In general, $\frac{d}{dt}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + V^{-1}B\vec{u}(t) = \Lambda\vec{\tilde{x}}(t) + \tilde{u}(t)$

☆ Then, $\vec{\tilde{x}}(t) = \vec{\tilde{x}}_h(t) + \vec{\tilde{x}}_n(t)$, where $\vec{\tilde{x}}_n(t) = \begin{bmatrix} \int_0^t \tilde{u}_1(\theta)\, e^{\lambda_1(t-\theta)} d\theta \\ \int_0^t \tilde{u}_2(\theta)\, e^{\lambda_2(t-\theta)} d\theta \end{bmatrix}$

Let $\tilde{B} = V^{-1}B$. $\vec{\tilde{u}} = V^{-1}B\vec{u} = \tilde{B}\vec{u} = \begin{bmatrix} \tilde{b}_{11}u_1(t) + \tilde{b}_{12}u_2(t) \\ \tilde{b}_{21}u_1(t) + \tilde{b}_{22}u_2(t) \end{bmatrix}$.

$B = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{b} \to \tilde{u}(t) = V^{-1}\vec{b}\, V_{in}(t) = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}(t) = \begin{bmatrix} 3/5 \\ -6/5 \end{bmatrix} V_{in}(t)$

$$\to \vec{\tilde{x}}_n = \begin{bmatrix} \int_0^t 3/5\, V_{in}(\theta)\, e^{-(t-\theta)} d\theta \\ \int_0^t -6/5\, V_{in}(\theta)\, e^{-6(t-\theta)} d\theta \end{bmatrix} \to \vec{x}_n = V\vec{\tilde{x}}_n = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{5}e^{-t}\int_0^t \cdots d\theta \\ -\frac{6}{5}e^{-6t}\int_0^t \cdots d\theta \end{bmatrix}$$

$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

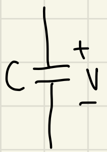
$$= \begin{bmatrix} \frac{3}{5}e^{-t}\int_0^t V_{in}(\theta)\, e^{\theta} d\theta + \frac{12}{5}e^{-6t}\int_0^t V_{in}(\theta)\, e^{6\theta} d\theta \\ \frac{6}{5}e^{-t}\int_0^t V_{in}(\theta)\, e^{\theta} d\theta - \frac{6}{5}e^{-6t}\int_0^t V_{in}(\theta)\, e^{6\theta} d\theta \end{bmatrix}$$

Now, for $V_{in}(t) = \begin{cases} 0V;\ t < 0 \\ 1V;\ t \geq 0 \end{cases}$. $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{x}_h(t) = \vec{0}$.

$$\vec{x}_n = \begin{bmatrix} 3/5\, e^{-t}(e^{t} - 1) + \frac{12}{5}e^{-6t}\cdot\frac{1}{6}(e^{6t} - 1) \\ 6/5\, e^{-t}(e^{t} - 1) - \frac{6}{5}e^{-6t}\cdot\frac{1}{6}(e^{6t} - 1) \end{bmatrix} = \begin{bmatrix} 3/5(1 - e^{-t}) + 2/5(1 - e^{-6t}) \\ 6/5(1 - e^{-t}) - 1/5(1 - e^{-6t}) \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 1 - 3/5\, e^{-t} - 2/5\, e^{-6t} \\ 1 - 6/5\, e^{-t} + 1/5\, e^{-6t} \end{bmatrix}} \to \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \vec{x}_n(t) = \vec{x}_n(t).$$

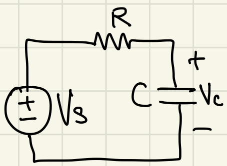
Inductors



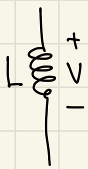
Capacitors: $\boxed{I(t) = C\frac{dV}{dt}(t)} \rightarrow$ "resists" a change in voltage

stores energy in an electric field, $E = \frac{1}{2}CV^2$, [F]

In DC (constant voltage), acts as an open circuit ($I = 0$)



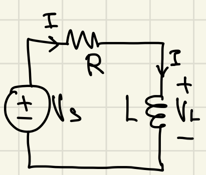
$V_c \sim e^{-t/RC}$, $RC = \tau \rightarrow V_c \sim e^{-t/\tau}$



Inductors: $\boxed{V(t) = L\frac{d}{dt}I(t)} \rightarrow$ "resists" a change in current

stores energy in a magnetic field, $E = \frac{1}{2}LI^2$, [H]

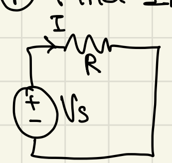
At constant current, acts as a short circuit ($V = 0$)



$V_L \sim e^{-t/\tau}$? $V_s(t<0) = 1V$, $V_s(t \geq 0) = 0V$.

Solve for $I_L(t)$ $(t \geq 0)$. $(I(t))$

① Find $I_L(0)$. For $t<0$, steady state. Then, inductor $\rightarrow$ wire.



$\underset{(t<0)}{I(t)} = \frac{V_s}{R} = \frac{1}{R} \rightarrow I_L(0) = \frac{1}{R}$

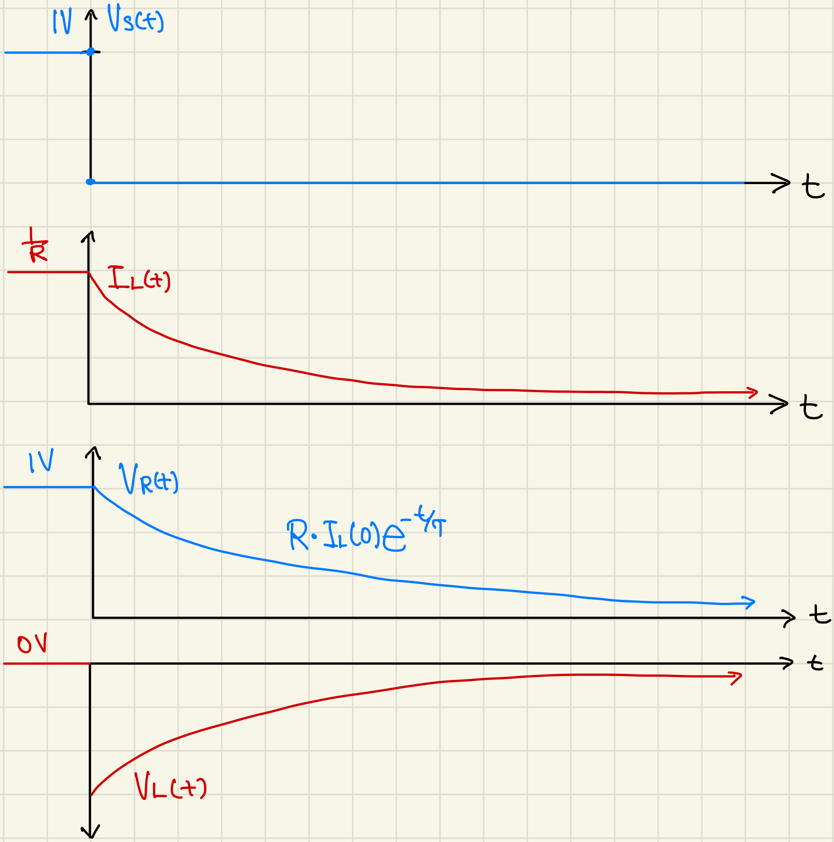
② Solve for $t \geq 0$. $V_L(t) = L\frac{d}{dt}I_L(t)$, $I_L(t) = \frac{V_s(t) - V_L(t)}{R}$

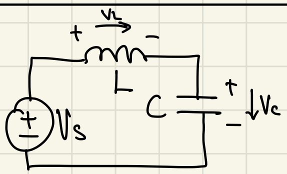
$\rightarrow I_L(t) = \frac{V_s(t)}{R} - \frac{L}{R}\frac{d}{dt}I_L(t)$. For $t \geq 0$, $\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t)$.

$\rightarrow I_L(t) = \frac{1}{R}e^{-\frac{R}{L}t} = I_L(0)e^{-t/\tau}$ ($\tau = L/R$, $I_L(0) = \frac{1}{R}$)

$$V_R(t) = R\cdot I_R(t) = R\cdot I_L(t) = R\cdot I_L(0)e^{-t/\tau} = e^{-t/\tau}$$

$$V_L(t) = \cancelto{0}{V_S(t)} - V_R(t) = -V_R(t) = -e^{-t/\tau}$$

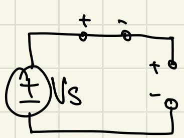




$$I_c(t) = C\frac{d}{dt}V_c(t),\quad V_L(t) = L\frac{d}{dt}I_L(t),\quad I_c(t) = I_L(t)$$

$$V_L(t) = V_S(t) - V_C(t) \qquad V_S = \begin{cases} 1V & ; t<0 \\ 0V & ; t\geq 0 \end{cases}$$

① Find initial condition. steady-state equivalance:



$V_c(0) = 1V$, $I_L(0) = 0A$

② Solve for $t \geq 0$: $V_s(t) = 0 \rightarrow V_L(t) = -V_c(t),\ I_L(t) = I_c(t)$

$\rightarrow L\frac{d}{dt}I_L(t) = -V_c(t),\ I_L(t) = C\frac{d}{dt}V_c(t)$

$\rightarrow \frac{d}{dt}I_L(t) = \frac{-1}{L}V_c(t),\ \frac{d}{dt}V_c(t) = \frac{1}{C}I_L(t) \rightarrow$ Matrix Form!

$\rightarrow \frac{d}{dt}\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & 0 \end{bmatrix}\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} \rightarrow$ diagonalize?

$\det(\lambda I - A) = 0 \rightarrow \det\left(\begin{bmatrix} \lambda & 1/L \\ -1/C & \lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{\frac{1}{LC}}\,j$

*Assume: $L = 1F,\ C = 1H.$ $\rightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \lambda_{1,2} = \pm j$

$\rightarrow$ Find vectors s.t. $A\vec{v}_1 = \lambda_1\vec{v}_1,\ A\vec{v}_2 = \lambda_2\vec{v}_2$

$\lambda_1 = j$: $(A - \lambda I)\vec{v}_1 = 0. \rightarrow \begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix}\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} j \\ 1 \end{bmatrix}$

$\lambda_2 = -j$: $Ak\vec{v}_2 = \lambda_2 k\vec{v}_2.$ $\vec{v}_2 = \begin{bmatrix} 1 \\ x \end{bmatrix} \rightarrow A\vec{v}_2 = \lambda_2\vec{v}_2 \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = -j\begin{bmatrix} 1 \\ x \end{bmatrix}$

$\rightarrow -x = -j,\ 1 = -jx \rightarrow x = j \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$

Transform coordinates: $\vec{x} = V\vec{\tilde{x}} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = \underset{\text{coordinate}}{\tilde{x}_1}\underset{\text{unit vector}}{\vec{v}_1} + \cdots + \tilde{x}_n V_n$

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d}{dt}\vec{x} \arrow[d, bend right, "V^{-1}"'] \\ \vec{\tilde{x}} \arrow[u, bend right, "V"'] \arrow[r, "V^{-1}AV"', "\hat{A}"] & \frac{d}{dt}\vec{\tilde{x}} \arrow[u, bend right, "V"'] \end{tikzcd}$$

found a way to express $\vec{x}$ in V-basis with $\vec{\tilde{x}}$.

$\frac{d}{dt}\vec{x}(t) = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{A}\vec{x}(t) \Rightarrow \frac{d}{dt}\vec{\tilde{x}}(t) = \overbrace{V^{-1}AV}^{\hat{A}}\vec{\tilde{x}} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}\vec{\tilde{x}}$

$\rightarrow \vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(0)e^{jt} \\ \tilde{x}_2(0)e^{-jt} \end{bmatrix}.$ $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 1/2 & j/2 \\ 1/2 & -j/2 \end{bmatrix}\begin{bmatrix} 0 \rightarrow I_L(0) \\ 1 \rightarrow V_c(0) \end{bmatrix}$

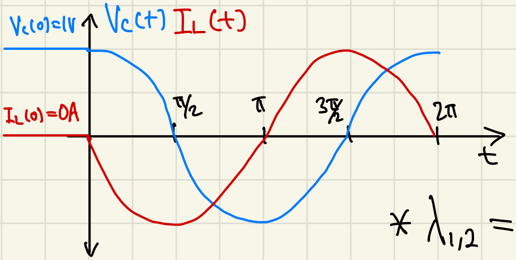
$\rightarrow \vec{\tilde{x}}(0) = \begin{bmatrix} j/2 \\ -j/2 \end{bmatrix} \Rightarrow \vec{\tilde{x}} = \begin{bmatrix} j/2\, e^{jt} \\ -j/2\, e^{-jt} \end{bmatrix} \rightarrow \vec{x} = V\vec{\tilde{x}} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}\begin{bmatrix} j/2\, e^{jt} \\ -j/2\, e^{-jt} \end{bmatrix}$

$\rightarrow \begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} j/2\, e^{jt} - j/2\, e^{-jt} \\ 1/2\, e^{jt} + 1/2\, e^{-jt} \end{bmatrix} \Rightarrow$ Euler's Formula $\Rightarrow I_L(t) = j/2\left(e^{jt} - e^{-jt}\right)$

$= j/2\left(\cancel{\cos t} + j\sin t - \cancel{\cos(-t)} - j\sin(-t)\right) = j/2\left(2j\sin t\right) = \underline{-\sin t}$

$\left(V_c(t) = \cos t\right)$

$\vec{x}(t) = \begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \rightarrow$ real functions! $\left(\left[\frac{rad}{s}\right] \circ t\right)$



Phase shifted by $\pi/2$ rad,

Energy moves between L and C

$* \ \lambda_{1,2} = \pm j\sqrt{\frac{1}{LC}}, \ \vec{x}(t) = \begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} -\sin(\frac{1}{\sqrt{LC}}t) \\ \cos(\frac{1}{\sqrt{LC}}t) \end{bmatrix}$

Phasors

$\frac{d}{dt}x(t) = \lambda x(t) + b u(t), \ u(t) = k \cdot e^{st}, \ s \neq \lambda.$

$$\rightarrow x(t) = \underbrace{\left(x(0) - \frac{bk}{s-\lambda}\right)e^{\lambda t}} + \underbrace{\frac{bk}{s-\lambda}e^{st}}$$

transient solution b/c of initial condition (annoying term)

form of $u(t)$! (nice term, steady-state solution)

Want transient part to disappear as $t \rightarrow \infty \Rightarrow \lambda < 0 : e^{\lambda t} \rightarrow 0$

$\Rightarrow x(t) \rightarrow \frac{bk}{s-\lambda}e^{st}$ (steady-state solution)

What about complex λs? $\lambda = \lambda_{re} + \lambda_{im} j \rightarrow e^{\lambda_{re} t} \cdot e^{j\lambda_{im} t}$

$\rightarrow e^{\lambda_{re} t} \cdot (\cos(\lambda_{im} t) + j\sin(\lambda_{im} t)) \Rightarrow$ if $\mathrm{Re}\{\lambda\} < 0 : e^{\lambda_r t} \rightarrow 0$

$\Rightarrow x(t) \rightarrow \frac{bk}{s-\lambda}e^{st}$ (steady-state solution)

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t)$, $\vec{u}(t) \sim e^{st}$, then assert that solutions are also $\sim e^{st}$ when $Re\{\lambda\}<0$ and $s \neq \lambda$ in steady-state.

$\vec{u}(t) = \vec{\tilde{u}}e^{st}$, where $\vec{\tilde{u}}$ is a vector of constants.

Assert: $\vec{x} = \vec{\tilde{x}}e^{st}$, where $\vec{\tilde{x}}$ is a vector of constants.

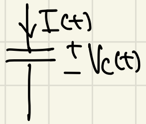
$$\frac{d}{dt}\vec{x}(t) = \vec{\tilde{x}} \cdot s e^{st} = A \cdot \vec{\tilde{x}} \cdot e^{st} + \vec{\tilde{u}} e^{st} \Rightarrow s \cdot \vec{\tilde{x}} = A\vec{\tilde{x}} + \vec{\tilde{u}}$$

$\rightarrow (sI-A)\vec{\tilde{x}} = \vec{\tilde{u}} \Rightarrow \vec{\tilde{x}} = (sI-A)^{-1}\vec{\tilde{u}}$ → system of linear equations!

($* \; s \neq \lambda \Rightarrow sI-A$ has no nullspace $\Rightarrow sI-A$ is invertible.)

$\rightarrow \vec{x}(t) = (sI-A)^{-1}\vec{\tilde{u}} \cdot e^{st}$ (solution to initial equation)

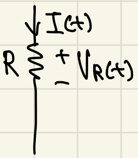
Can we use this to analyze circuits?

ex)


$I(t) = C\frac{d}{dt}V(t)$. Assert: $I(t) = \tilde{I}e^{st}$, $V(t) = \tilde{V}e^{st}$.

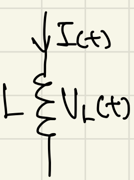
$\rightarrow \frac{V(t)}{I(t)} = \frac{\tilde{V}}{\tilde{I}}$! $\tilde{I}e^{st} = C \cdot \frac{d}{dt}(\tilde{V}e^{st}) \rightarrow \tilde{I}e^{st} = sC(\tilde{V}e^{st})$

$\Rightarrow \tilde{I} = sC\tilde{V} \Rightarrow \underline{\frac{\tilde{V}}{\tilde{I}} = \frac{1}{sC}}$. → capacitor s-impedance



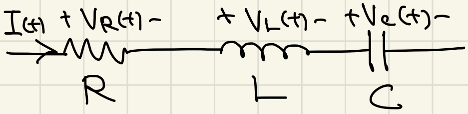
$V(t) = \tilde{V}e^{st}$, $I(t) = \tilde{I}e^{st}$. $V = IR \rightarrow \tilde{V}e^{st} = \tilde{I}e^{st} \cdot R$

$\rightarrow \underline{\frac{\tilde{V}}{\tilde{I}} = R}$ → resistor s-impedance



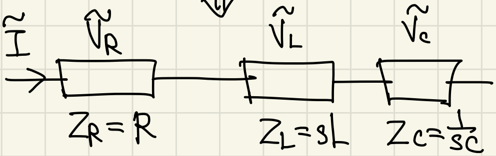
$V(t) = L\frac{d}{dt}I(t) \rightarrow \tilde{V}e^{st} = L \cdot s \cdot \tilde{I}e^{st}$

$\rightarrow \underline{\frac{\tilde{V}}{\tilde{I}} = sL}$ → inductor s-impedance



$V_R = \tilde{V}_R e^{st}$, $V_L = \tilde{V}_L e^{st}$, $V_c = \tilde{V}_c e^{st}$, $I = \tilde{I} e^{st}$

$\Downarrow$



$\tilde{V}_R = \tilde{I} \cdot Z_R$, $\tilde{V}_L = \tilde{I} \cdot Z_L$, $\tilde{V}_c = \tilde{I} \cdot Z_c$

For sinusoids, $u(t) = U\cos(\omega t + \phi) = U \dfrac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}$

$$= \underbrace{\underbrace{\frac{Ue^{j\phi}}{2}}_{\tilde{u}} \underbrace{e^{j\omega t}}_{e^{s_1 t}}} + \underbrace{\underbrace{\frac{Ue^{-j\phi}}{2}}_{\bar{\tilde{u}}} \underbrace{e^{-j\omega t}}_{e^{s_2 t}}} \quad (s_1 = j\omega,\ s_2 = -j\omega)$$

$\rightarrow u(t) = \tilde{u} e^{s_1 t} + \bar{\tilde{u}} e^{s_2 t}$.

always complex conjugates (b/c u(t) is real)

Since circuit is linear, use superposition to solve for each term.

$\Rightarrow$ $\underline{\vec{x}(t) = \vec{\tilde{x}}_1 e^{s_1 t} + \vec{\tilde{x}}_2 e^{s_2 t}}$ is a solution form.

① $s_1 = j\omega$: $M_1 = s_1 I - A(s_1) = j\omega I - A(j\omega)$

$\rightarrow M_1 \vec{\tilde{x}}_1 = \vec{\tilde{u}} \rightarrow \vec{\tilde{x}}_1 = M_1^{-1} \cdot \vec{\tilde{u}}$

$\begin{bmatrix} \tilde{I} \\ \tilde{V} \end{bmatrix} \rightarrow$ elements currents voltages

M_1^{-1}: circuit topology

$\vec{\tilde{u}}$: independent source

② $s_2 = -j\omega$: $M_2 = s_2 I - A(s_2) = -j\omega I - A(-j\omega) = -j\omega I - \overline{A(j\omega)} = \overline{M_1}$

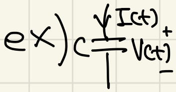
$\rightarrow M_2 \vec{\tilde{x}}_2 = \bar{\vec{\tilde{u}}} \rightarrow \vec{\tilde{x}}_2 = M_2^{-1} \cdot \bar{\vec{\tilde{u}}} = \overline{M_1}^{-1} \bar{\vec{\tilde{u}}} = \overline{(M_1^{-1} \vec{\tilde{u}})} = \overline{\vec{\tilde{x}}_1}$

So $\vec{x}(t) = \underbrace{\vec{\tilde{x}}_1 e^{j\omega t} + \overline{\vec{\tilde{x}}_1} e^{-j\omega t}}_{\text{(complex conjugates)}}$ (real) $\rightarrow$ only need to solve for one of $\vec{\tilde{x}}_1$ or $\vec{\tilde{x}}_2$!

So, all solutions: $\vec{V}(t) = \tilde{V}e^{j\omega t} + \overline{\tilde{V}}e^{-j\omega t}$, $\vec{I}(t) = \tilde{I}e^{j\omega t} + \overline{\tilde{I}}e^{-j\omega t}$.

For sinusoids, $\tilde{V}$ and $\tilde{I}$ are phasors. (functions of $s = j\omega$)

Now, s-impedances are just impedances.

ex) 

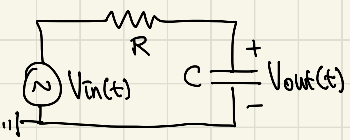
$V(t) = V_0\cos(\omega t + \phi) \to V(t) = \underbrace{\frac{V_0}{2}e^{j\phi}}_{\tilde{V}} \cdot e^{j\omega t} + \underbrace{\frac{V_0}{2}e^{-j\phi}}_{\overline{\tilde{V}}} \cdot e^{-j\omega t}$

$I(t) = C\frac{d}{dt}V(t) = C\frac{d}{dt}\left(\tilde{V}e^{j\omega t} + \overline{\tilde{V}}e^{-j\omega t}\right)$

$= \underbrace{j\omega C\tilde{V}}_{\tilde{I}}e^{j\omega t} \underbrace{- j\omega C\overline{\tilde{V}}}_{\overline{\tilde{I}}}e^{-j\omega t} = \tilde{I}e^{j\omega t} + \overline{\tilde{I}}e^{-j\omega t} \to \tilde{I} = j\omega C\tilde{V}$

$\to \frac{\tilde{V}}{\tilde{I}} = \frac{1}{j\omega C}$

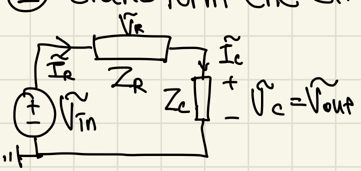
ex) RC circuit. $V_{in}(t) = V_{in}\cos(\omega t + \phi)$. $V_{out}(t)$?



① write sources as exponentials (into phasor domain)

$V_{in}(t) = \underbrace{\frac{V_{in}}{2}e^{j\phi}}_{\tilde{V}_{in}}e^{j\omega t} + \underbrace{\frac{V_{in}}{2}e^{-j\phi}}_{\overline{\tilde{V}_{in}}}e^{-j\omega t}$

② transform the circuit to phasor domain



$Z_R = R$, $Z_C = \frac{1}{j\omega C}$. (s-impedances, $s = j\omega$)

③ write down circuit equations.

$\tilde{V}_R = \tilde{I}_R \cdot Z_R$, $\tilde{V}_c = \tilde{I}_c \cdot Z_c$, $\tilde{I}_R = \tilde{I}_c$, $\tilde{V}_R = \tilde{V}_{in} - \tilde{V}_c$, $\tilde{V}_c = \tilde{V}_{out}$

④ solve the circuit. $\tilde{V}_{out} = \tilde{V}_{in}\frac{Z_c}{Z_R + Z_c} = \tilde{V}_{in}\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}\tilde{V}_{in}$

$\to \tilde{V}_{out}(j\omega) = \frac{1}{1 + j\omega RC} \cdot \tilde{V}_{in}(j\omega)$ $\quad \tilde{V}_{in} \longrightarrow \boxed{H(j\omega)} \longrightarrow \tilde{V}_{out}$ $\quad \underbrace{\frac{\tilde{V}_{out}(j\omega)}{\tilde{V}_{in}(j\omega)}}_{H(j\omega)}$ $\to$ transfer function

$$H_{LP}(jw) = \frac{1}{1+jwRC} = \frac{1-jwRC}{1+(wRC)^2}$$

$$|H_{LP}(jw)| = \frac{1}{|1+jwRC|} = \frac{1}{\sqrt{1+(wRC)^2}}, \ \measuredangle H_{LP}(jw) = -\text{atan2}(wRC, 1)$$

$$\tilde{V}_{out} = |\tilde{V}_{out}| e^{j\measuredangle \tilde{V}_{out}}, \ \tilde{V}_{in} = |\tilde{V}_{in}| e^{j\measuredangle \tilde{V}_{in}}$$

$$H(jw) = |H(jw)| e^{\measuredangle H(jw)}$$

$$\tilde{V}_{out} = |\tilde{V}_{out}| e^{j\measuredangle \tilde{V}_{out}} = \tilde{V}_{in} \cdot H(jw) = |\tilde{V}_{in}| e^{j\measuredangle \tilde{V}_{in}} \cdot |H(jw)| e^{j\measuredangle H(jw)}$$

$$= \underbrace{|H(jw)||\tilde{V}_{in}|} \cdot e^{j(\underbrace{\measuredangle H(jw) + \measuredangle \tilde{V}_{in}})}$$

$$|\tilde{V}_{out}| = |H(jw)| \cdot |\tilde{V}_{in}|, \ \measuredangle \tilde{V}_{out} = \measuredangle H(jw) + \measuredangle \tilde{V}_{in}$$

---

⑤ convert back to time domain

$$V_{out}(t) = \tilde{V}_{out} e^{jwt} + \overline{\tilde{V}_{out}} e^{-jwt}$$

$$= |\tilde{V}_{out}| e^{j\measuredangle \tilde{V}_{out}} \cdot e^{jwt} + |\tilde{V}_{out}| e^{-j\measuredangle \tilde{V}_{out}} \cdot e^{-jwt}$$

$$= |\tilde{V}_{out}| \left(e^{j(\measuredangle \tilde{V}_{out} + wt)} + e^{-j(\measuredangle \tilde{V}_{out} + wt)}\right)$$

$$= 2 \cdot |\tilde{V}_{out}| \cos(wt + \measuredangle \tilde{V}_{out})$$

similarly, $V_{in}(t) = 2|\tilde{V}_{in}| \cos(wt + \measuredangle \tilde{V}_{in})$

$$= 2 \frac{1}{\sqrt{1+(wRC)^2}} \frac{V_{in}}{2} \cos(\ ) = \frac{V_{in}}{\sqrt{1+(wRC)^2}} \cos(wt + \phi + \measuredangle H_{LP}(jw))$$

$\measuredangle H_{LP}(jw) \downarrow -\text{atan2}(wRC, 1)$

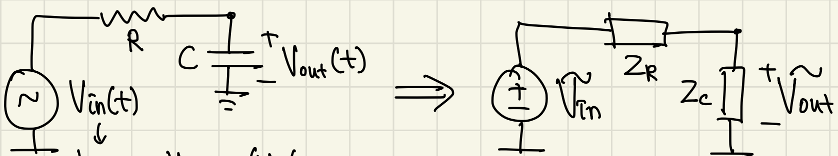
Filters

Objective: Find a steady-state solution of the system in response to sinusoidal inputs ($\lambda_r < 0$).

→ sine waves allows transform of differential eqations to linear equations (phasor domain)

Phasor analysis: exp → linear → exp

ex1) Low-Pass Filter



$V_{in}(t) = V_{in}\cos(\omega_{in}t + \phi)$

$\tilde{V}_{out} = ?$

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_C}{Z_R + Z_C} = H_{LP}(j\omega) \rightarrow \text{transfer function}$$

cutoff frequency

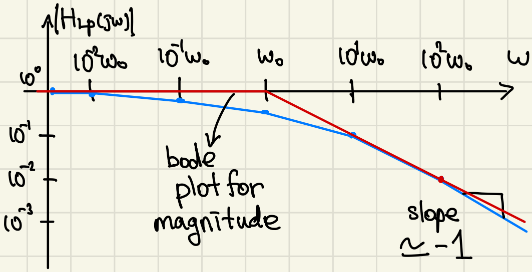
$$H_{LP}(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\frac{\omega}{\omega_0}} \quad \left(\omega_0 = \frac{1}{RC}\right)$$

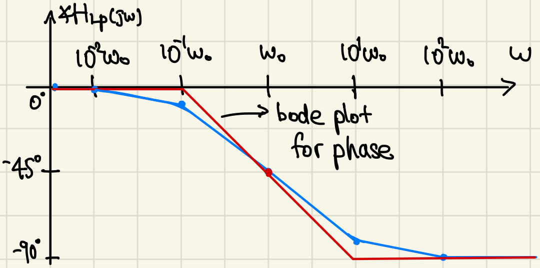
$$\rightarrow \tilde{V}_{out} = H_{LP}(j\omega_{in}) \cdot \tilde{V}_{in}.$$

$$|H_{LP}(j\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_0})^2}} = \begin{cases} \omega \gg \omega_0 \rightarrow |H_{LP}| \simeq 0 \\ \omega \ll \omega_0 \rightarrow |H_{LP}| \simeq 1 \end{cases}$$

$$\measuredangle H_{LP}(j\omega) = -\operatorname{atan2}\left(\frac{\omega}{\omega_0}, 1\right) = \begin{cases} \omega \gg \omega_0 \rightarrow \measuredangle H_{LP} \simeq -\pi/2 \\ \omega \ll \omega_0 \rightarrow \measuredangle H_{LP} \simeq 0 \end{cases}$$

ω	$H_{LP}(j\omega)$	$\lvert H_{LP}(j\omega)\rvert$	$\measuredangle H_{LP}(j\omega)$
$\omega \ll \omega_0$	$\simeq 1$	$\simeq 1$	$\simeq 0$
$0.1\,\omega_0$	$1/(1+0.1j)$	0.995	$\simeq -6^\circ$
ω_0	$1/(1+j)$	$\frac{1}{\sqrt{2}} \simeq 0.71$	$\simeq -45^\circ$
$10\,\omega_0$	$1/(1+10j)$	0.1	$\simeq -84^\circ$
$\omega \gg \omega_0$	$\simeq j\frac{\omega_0}{\omega}$	$\frac{\omega_0}{\omega}$	$\simeq -90^\circ$





$$V_{out}(t) = \tilde{V}_{out} e^{j\omega t} + \overline{\tilde{V}_{out}} e^{-j\omega t}$$

$$= 2|\tilde{V}_{out}|\cos(\omega t + \measuredangle \tilde{V}_{out})$$

$$\tilde{V}_{out} = H(j\omega)\,\tilde{V}_{in}$$

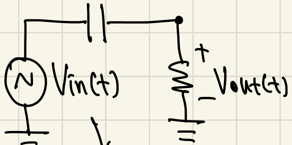
$$\rightarrow |\tilde{V}_{out}| e^{j\measuredangle V_{out}} = |H(j\omega)||\tilde{V}_{in}| \cdot e^{j(\measuredangle \tilde{V}_{in} + \measuredangle H(j\omega))} \quad (\omega = \omega_{in})$$

$$\Rightarrow V_{out}(t) = 2(|H(j\omega)||\tilde{V}_{in}|)\cos(\omega t + \measuredangle \tilde{V}_{in} + \measuredangle H(j\omega)) \quad (\omega = \omega_{in})$$

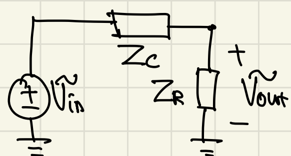
for input $V_{in}(t) = V_{in} \cdot \cos(\omega_{in} t + \phi)$

$$\rightarrow V_{out}(t) = 2|H(j\omega)||\tilde{V}_{in}|\cos(\omega_{in} t + \phi + \measuredangle H(j\omega))$$

ex2) High-Pass Filter



$\Rightarrow$



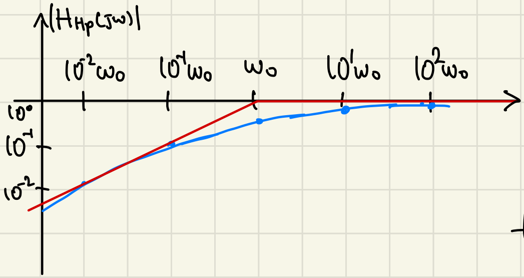
$V_{in}(t) = V_{in}\cos(\omega t + \phi)$

$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + 1/j\omega C} = H_{HP}(j\omega)$$

$$H_{HP}(j\omega) = \frac{j\omega RC}{1 + j\omega RC}, \quad \omega_0 = \frac{1}{RC}$$

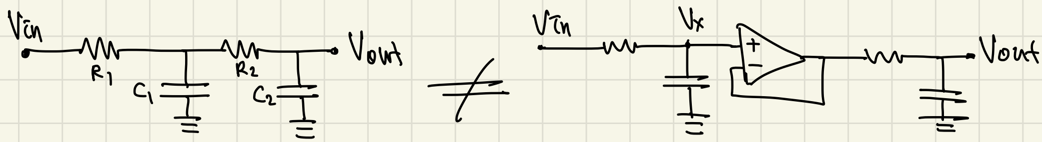
$$= \frac{j\frac{\omega}{\omega_0}}{1 + j\frac{\omega}{\omega_0}} = \frac{1}{1 - j\frac{\omega_0}{\omega}}$$

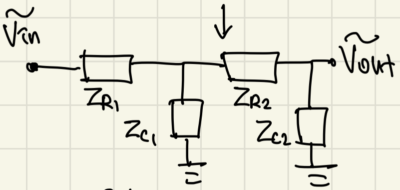
$$|H_{HP}(j\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega_0}{\omega})^2}} = \begin{cases} \omega \gg \omega_0 \rightarrow |H_{HP}| \simeq 1 \\ \omega \ll \omega_0 \rightarrow |H_{HP}| \simeq O(\frac{\omega}{\omega_0}) \end{cases}$$

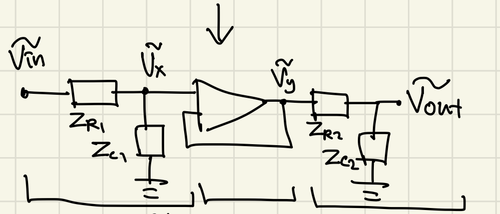


How do we cascade these circuits to build more complex transfer functions?

$\Rightarrow$ Circuit blocks should not load (take current from each other) in order to preserve transfer functions!







$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H(j\omega)$$

$$H_1(j\omega) = \frac{\tilde{V}_x}{\tilde{V}_{in}} \quad H_B \approx 1 \quad H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_y}$$

$$H_{Tot}(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_y} \cdot \frac{\tilde{V}_y}{\tilde{V}_x} \cdot \frac{\tilde{V}_x}{\tilde{V}_{in}} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

$$H_{Tot}(j\omega) = \frac{1}{1+j\frac{\omega}{\omega_{o_1}}} \cdot \cancel{\times} \cdot \frac{1}{1+j\frac{\omega}{\omega_{o_2}}} \quad (\omega_{o_1} = \frac{1}{R_1C_1}, \omega_{o_2} = \frac{1}{R_2C_2}$$

$$\tilde{V}_{in} \rightarrow \boxed{H_1(j\omega)} \rightarrow \cdots \rightarrow \boxed{H_n(j\omega)} \rightarrow \tilde{V}_{out}$$

$$H_{Tot}(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \prod H_i(j\omega) = \prod |H_i(j\omega)| e^{j(\sum \measuredangle H_i(j\omega))}$$

$$\rightarrow V_{out}(t) = |H_{Tot}(j\omega)| \, 2|\tilde{V}_{in}| \cos(\omega t + \measuredangle \tilde{V}_{in} + \measuredangle H_{Tot}(j\omega))$$

Design

V_{in} has many components:

	frequency	magnitude	
signal	600 Hz	1 mV	→ desired
AC	60 Hz	10 mV	→ interference
fluorescent light	60 kHz	20 mV	

design goal: attenuate interference (AC, fluorescent light) by 100X + keep signal

$$V_{in}(t) = V_{AC}\cos(\omega_{AC}t + \phi_{AC}) + V_{sig}\cos(\omega_s t + \phi_s) + V_{flu}\cos(\omega_f t + \phi_f)$$

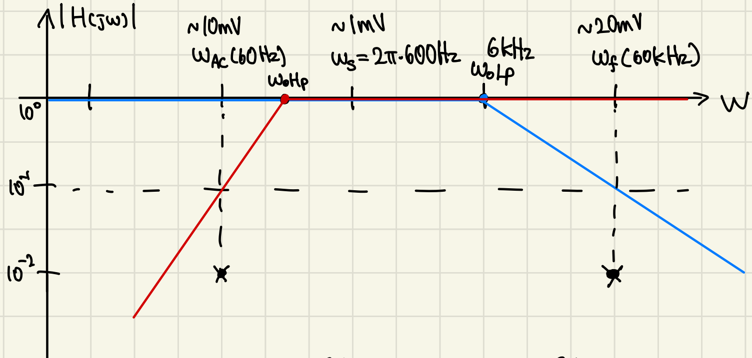
$$(\omega_{AC} = 2\pi \cdot 60\,Hz = 377\,\tfrac{rad}{s},\ \omega_s = 2\pi \cdot 600\,Hz,\ \omega_f = 2\pi \cdot 60\,kHz)$$

Strategy: $\tilde{V}_{in} \rightarrow \boxed{H(j\omega)} \rightarrow \tilde{V}_{out}$, but $\tilde{V}_{in} = \tilde{V}_{AC} + \tilde{V}_{sig} + \tilde{V}_{flu}$. superposition!

$$\rightarrow \tilde{V}_{out}(t) = \tilde{V}_{AC} \cdot H(j\omega_{AC}) + \tilde{V}_{sig} \cdot H(j\omega_s) + \tilde{V}_{flu} \cdot H(j\omega_f)$$

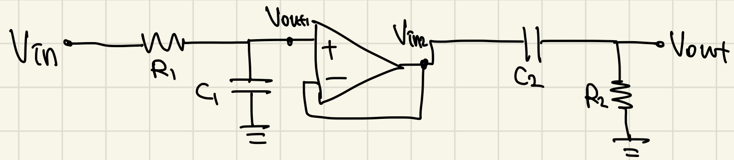
$$\Rightarrow V_{out}(t) = |H(j\omega_{AC})| \cdot V_{AC} \cdot \cos(\omega_{AC}t + \phi_{AC} + \angle H(j\omega_{AC}))$$
$$+ |H(j\omega_s)| \cdot V_{sig} \cdot \cos(\omega_s t + \phi_s + \angle H(j\omega_s))$$
$$+ |H(j\omega_f)| \cdot V_{flu} \cdot \cos(\omega_f t + \phi_f + \angle H(j\omega_f))$$

design goal: $|H(j\omega_{AC})| = |H(j\omega_f)| \leq \frac{1}{100}$, $|H(j\omega_s)| = 1$.



$$\tilde{V}_{in} \rightarrow \boxed{H_{Lp}(j\omega)} \xrightarrow{\tilde{V}_{out_1}} \boxed{\text{Buffer}} \xrightarrow{\tilde{V}_{in_2}} \boxed{H_{Hp}(j\omega)} \rightarrow \tilde{V}_{out}$$

- $\omega_1 = \frac{1}{R_1C_1}$
- $\omega_2 = \frac{1}{R_2C_2}$



$$H_{Lp}(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_1}}$$

$$H_{Hp}(j\omega) = \frac{1}{1 - j\frac{\omega_2}{\omega}}$$

$$\Rightarrow H_{Tot}(j\omega) = \underbrace{\frac{\tilde{V}_{out}}{\tilde{V}_{in_2}}} \cdot \underbrace{\frac{\tilde{V}_{in_2}}{\tilde{V}_{out_1}}} \cdot \underbrace{\frac{\tilde{V}_{out_1}}{\tilde{V}_{in}}} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = H_{Hp}(j\omega) \cdot H_{Lp}(j\omega)$$

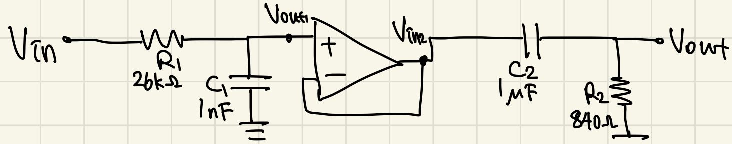
Compromise: want to attenuate interference w/o attenuating signal.

$$\omega_{oHp} = \omega_2 = \sqrt{\omega_{AC} \cdot \omega_s} = \frac{1}{R_2C_2}, \quad \omega_{oLp} = \omega_1 = \sqrt{\omega_s \cdot \omega_f} = \frac{1}{R_1C_1}$$

$$\omega_{oHp} = \sqrt{2\pi \cdot 60 \cdot 2\pi 600} \simeq 2\pi \cdot 190\text{ Hz}, \quad \omega_{oLp} = 2\pi \cdot 6\text{kHz}$$

Pick a reasonable capacitor. $\rightarrow C_1 = 1\text{nF} \rightarrow R_1 \simeq 26\text{k}\Omega$

$C_2 = 1\mu\text{F} \rightarrow R_2 = 840\,\Omega \rightarrow$ both reasonable resistances.



Check: evaluate $H(j\omega)$ at ω_{AC}, ω_s, and ω_f.

ω	$\lvert H_{Lp}(j\omega)\rvert$	$\lvert H_{Hp}(j\omega)\rvert$	$\lvert H(j\omega)\rvert$	$V_{in}\lvert H(j\omega)\rvert = V_{out}$
$2\pi \cdot 60$Hz	$\simeq 1$	$\simeq 0.3$	$\simeq 0.3$	$10\text{mV} \cdot 0.3 = 3\text{mV}$
$2\pi \cdot 600$Hz	$\simeq 1$	$\simeq 0.95$	$\simeq 0.95$	$1\text{mV} \cdot 0.95 = 0.95\text{mV}$
$2\pi \cdot 60$kHz	$\simeq 0.1$	$\simeq 1$	$\simeq 0.1$	$20\text{mV} \cdot 0.1 = 2\text{mV}$

wanted 0.01 for ω_{AC}, ω_f

but only got 0.3, 0.1.

$\rightarrow$ not good enough

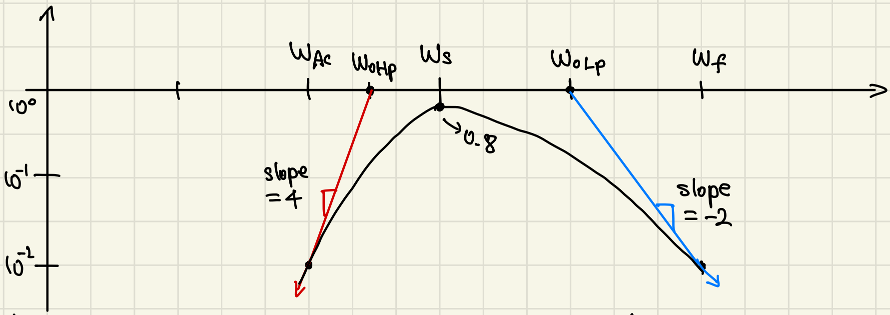
Keep going . . . ?

$$\rightarrow \boxed{H_{LP}(j\omega)} \rightarrow \boxed{\text{Buffer}} \rightarrow \boxed{H_{HP}(j\omega)} \rightarrow \boxed{\text{Buffer}} \rightarrow \boxed{H_{LP}(j\omega)} \rightarrow \cdots$$

$$H_{Tot}(j\omega) = H_{LP}^{n}(j\omega) \cdot H_{HP}^{m}(j\omega) \quad (n \text{ low-passes, } m \text{ high-passes})$$

$$|H_{tot}(j\omega_{AC})| = \frac{1}{100} = 1^n \cdot (0.3)^m \rightarrow m \simeq 4$$

$$|H_{tot}(j\omega_f)| = \frac{1}{100} = (0.1)^n \cdot 1^m \rightarrow n \simeq 2$$



$$|H_{tot}(j\omega_s)| = 1^n \cdot 0.95^m = 0.95^4 \simeq 0.8 \quad (\text{ok, not great})$$

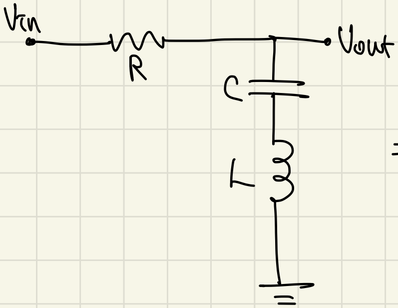
$$H_{Tot}(j\omega) = \frac{\left(j\frac{\omega}{\omega_{oHP}}\right)^4}{\left(1+j\frac{\omega}{\omega_{oHP}}\right)^4 \left(1+j\frac{\omega}{\omega_{oLP}}\right)^2}.$$

(In general, $H(j\omega) = K \dfrac{(j\omega)^{N_{zo}}\left(1+j\frac{\omega}{\omega_{z1}}\right)\cdots\left(1+j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{po}}\left(1+j\frac{\omega}{\omega_{p1}}\right)\cdots\left(1+j\frac{\omega}{\omega_{pn}}\right)}$

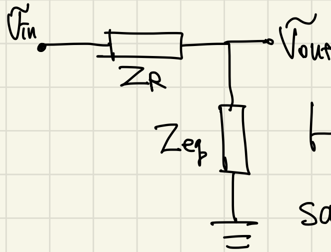
ω_{zn} - zeros, ω_{pn} - poles)

What if desired signal is at 100 Hz?

→ need a different filter (inductors?)



$\Rightarrow$



$Z_{eq} = j(\omega L - \frac{1}{\omega C})$

$H(j\omega) = \dfrac{Z_{eq}}{Z_{eq} + Z_R}$

say $Z_{eq}(j\omega_0) = 0$ $(\omega_0 = \frac{1}{\sqrt{LC}})$

$\rightarrow |H(j\omega_0)| = \left|\dfrac{Z_{eq}(j\omega_0)}{Z_{eq}(j\omega_0) + Z_R}\right| = \left|\dfrac{0}{Z_R}\right| = 0.$

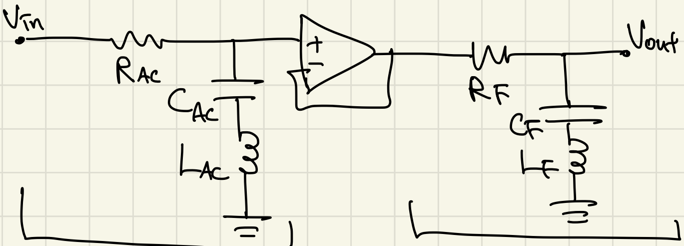
$H(j\omega) = \dfrac{j(\omega L - \frac{1}{\omega C})}{R + j(\omega C - \frac{1}{\omega C})}.$

→ set $\omega_0 = \omega_{AC} = 2\pi \cdot 60\,Hz$

→ $C = 100\mu F$, $L = 70\,mH$

$|H(j\omega_0)| = 0$, $|H(j \cdot 2\pi \cdot 55Hz)| \simeq |H(j \cdot 2\pi \cdot 65Hz)| \simeq 0.5$

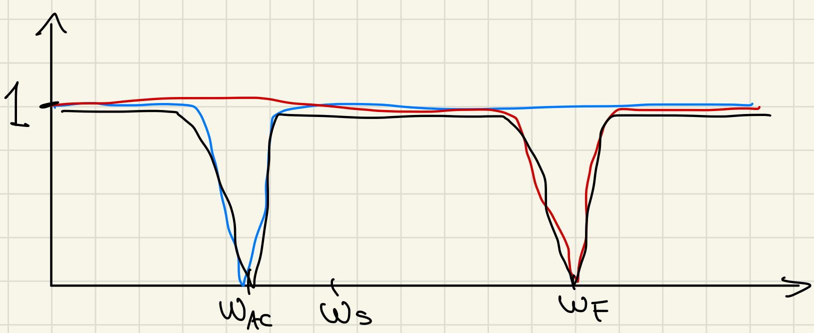
→ super sharp attenuation!



$\omega_{0AC} = \dfrac{1}{\sqrt{L_{AC} C_{AC}}}$ $\qquad \omega_{0F} = \dfrac{1}{\sqrt{L_F C_F}}$

R − LC gives sharp attenuation

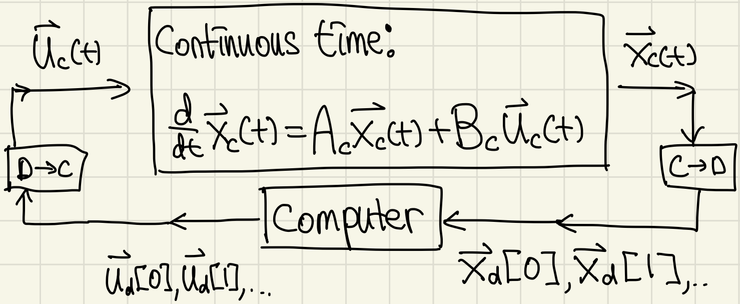
LC − R gives sharp peak

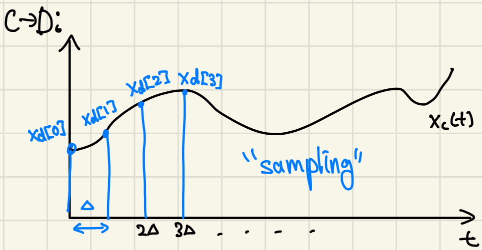


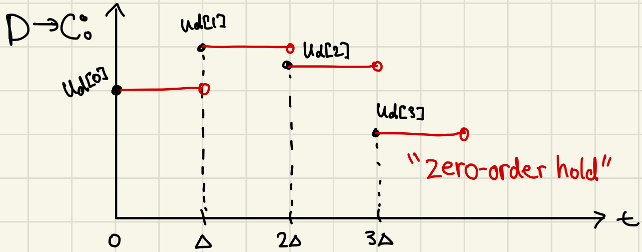
Control: What to do?

Control & Inference blocks are algorithms in discrete time.

Rest of the system flows in continuous time. How to connect?







$\vec{x}_d[i] = \vec{x}_c(i\Delta)$

$\vec{u}_c(t) = \vec{u}_d[i] \quad t \in [i\Delta, (i+1)\Delta)$

$$\vec{u}_d[i] \; \underbrace{\boxed{\to \text{d/c} \xrightarrow{\Delta} \frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_c(t) \to \text{c/d} \xrightarrow{\Delta}}}_{\vec{x}_d[i+1] = A_d\vec{x}_d[i] + B_d\vec{u}_d[i]} \; \vec{x}_d[i]$$

→ recurrence relation from sample to sample

⇒ How to find A_d and B_d, given A_c, B_c, and Δ?

i.e. given $\vec{x}_d[i]$ and $\vec{u}_d[i]$, what is $\vec{x}_d[i+1]$?

$\vec{x}_d[i] = \vec{x}_c(i\Delta)$, $\vec{x}_d[i+1] = \vec{x}_c((i+1)\Delta)$ from sampling equation

$\vec{x}_d[i+1]$ is the solution of $\frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_c(t)$ at $t = (i+1)\Delta$

from initial condition $\vec{x}_c(i\Delta) = \vec{x}_d[i]$ at $t_0 = i\Delta$. $\rightarrow \vec{u}_c(t) = \vec{u}_d[i]$.

ex) scalar system: $\frac{d}{dt}x_c(t) = \lambda x_c(t) + b u_d[i]$, $x_c(t_0) = x_d[i]$

$$\rightarrow x_c(t) = e^{\lambda(t-t_0)}x_c(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} b u_d[i]\, d\tau, \quad t_0 = i\Delta,\ t = (i+1)\Delta$$

$$\rightarrow \underbrace{x_c((i+1)\Delta)}_{x_d[i+1]} = \underbrace{e^{\lambda(\Delta)}}_{A_d}\underbrace{x_c(i\Delta)}_{x_d[i]} + \underbrace{\underbrace{\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right)}_{\text{when } \lambda=0,\ \Delta}}_{B_d} \cdot b\underbrace{u_d[i]}_{u_d[i]}$$

$$\rightarrow A_c = \lambda,\ B_c = b \Rightarrow A_d = e^{\lambda\Delta},\ B_d = \begin{cases} \frac{e^{\lambda\Delta}-1}{\lambda}b : \lambda \neq 0 \\ b\Delta : \lambda = 0 \end{cases}$$

ex) Vector system: $\frac{d}{dt}\vec{x}_c(t) = A_c\vec{x}_c(t) + B_c\vec{u}_d[i]$, $t_0 = i\Delta$, $t = (i+1)\Delta$

$$\vec{y}_c = V^{-1}x_c \rightarrow \frac{d}{dt}\vec{y}_c(t) = V^{-1}\frac{d}{dt}\vec{x}_c(t) = V^{-1}A_c\vec{x}_c(t) + V^{-1}B_c\vec{u}_d[i]$$

$$\rightarrow \frac{d}{dt}\vec{y}_c(t) = \underbrace{V^{-1}A_cV}_{\Lambda}\vec{y}_c(t) + \underbrace{V^{-1}B_c\vec{u}_d[i]}_{\vec{b}} \rightarrow \frac{d}{dt}y_{ck}(t) = \lambda_k y_{ck}(t) + b_k$$

$$\rightarrow y_{dk}[i+1] = e^{\lambda_k\Delta}y_{dk}[i] + \frac{e^{\lambda_k\Delta}-1}{\lambda_k}(V^{-1}B_c\vec{u}_d[i])_k, \quad k \in [1, n]$$

$$\rightarrow \vec{y}_d[i+1] = \begin{bmatrix} e^{\lambda_1\Delta} & & \\ & \ddots & \\ & & e^{\lambda_n\Delta} \end{bmatrix}\vec{y}_d[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}(V^{-1}B_c\vec{u}_d[i])$$

$$\rightarrow \vec{x}_d[i+1] = \underbrace{V\begin{bmatrix} e^{\lambda_1\Delta} & & \\ & \ddots & \\ & & e^{\lambda_n\Delta} \end{bmatrix}V^{-1}}_{A_d}\vec{x}_d[i] + \underbrace{V\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}V^{-1}B_c}_{B_d}\vec{u}_d[i]$$

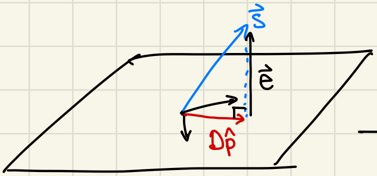
System Identification:

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + B_d \vec{u}_d[i] \quad \text{(can drop subscript } d\text{)}$$

Can we learn the entries of A_d & B_d by observing input sequence $\vec{u}_d[0], \vec{u}_d[1], \ldots$ and resulting sequence $\vec{x}_d[0], \vec{x}_d[1] \ldots$? → Yes, least squares.

$\vec{S} = D\vec{p}$, $\underset{\text{(measurments)}}{\vec{S} \in \mathbb{R}^l}$, $\underset{\text{(unknown)}}{\vec{p} = \mathbb{R}^q}$, $D \in \mathbb{R}^{l \times q}$ (known matrix), mostly $q < l$.

→ $\vec{S} = D\vec{p} + \vec{e}$, find $\hat{p}$ s.t. $D\hat{p}$ is as close to $\vec{S}$ as possible (minimize $\|\vec{e}\|$)



$D\hat{p} \in \text{span}(D)$

$\|\vec{e}\|$ is minimized when $\vec{e} \perp$ columnspace(D)

→ $\vec{d}_1^T \vec{e} = 0, \ldots, \vec{d}_q^T \vec{e} = 0$ where $D = [\vec{d}_1, \ldots, \vec{d}_q]$

→ $D^T \vec{e} = 0$ → $D^T(\vec{S} - D\hat{p}) = 0$ → $D^T D \hat{p} = D^T \vec{S}$

→ if $D^T D$ is invertible, $\hat{p} = (D^T D)^{-1} D^T \vec{S}$.

ex) Scalar case: $x[i+1] = \lambda x[i] + bu[i] + \underbrace{e[i]}_{\text{error/disturbance}}$

$$\begin{cases} x[1] = \lambda x[0] + bu[0] + e[0], \quad x[2] = \lambda x[1] + bu[1] + e[1] \\ \cdots \; x[l] = \lambda x[l-1] + bu[l-1] + e[l-1] \end{cases}$$

$$\underbrace{\begin{bmatrix} x_0 & u_0 \\ x_1 & u_1 \\ \vdots & \\ x_{l-1} & u_{l-1} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{l-1} \end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix}}_{\vec{S}}$$

if $D^T D$ is invertible, use L.S. $\hat{p} = (D^T D)^{-1} D^T \vec{S}$

→ $\hat{p} = \begin{bmatrix} \hat{\lambda} \\ \hat{b} \end{bmatrix}$ → best fit ex) Vector case: $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{e}[i]$

$$\begin{cases} \vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] + \vec{e}[0], \cdots \vec{x}[l] = A\vec{x}[l-1] + B\vec{u}[l-1] + \vec{e}[l-1] \end{cases}$$

transpose: $\vec{x}[0]^T A^T + \vec{u}[0]^T B^T + \vec{e}[0]^T = \vec{x}[1]^T, \ldots$

$$\vec{x}[l-1]^T A^T + \vec{u}[l-1]^T B^T + \vec{e}[l-1]^T = \vec{x}[l]^T$$

$$\boxed{\begin{array}{l} \vec{e} \in \mathbb{R}^n \\ \vec{x} \in \mathbb{R}^n, \vec{u} \in \mathbb{R}^m \\ \to A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \end{array}}$$

$$\to \underbrace{\begin{bmatrix} \vec{x}[0]^T & \vec{u}[0]^T \\ \vec{x}[1]^T & \vec{u}[1]^T \\ \vdots & \\ \vec{x}[l-1]^T & \vec{u}[l-1]^T \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}_{(n+m)\times n} + \underbrace{\begin{bmatrix} \vec{e}[0]^T \\ \vec{e}[1]^T \\ \vdots \\ e[l-1]^T \end{bmatrix}}_{\begin{bmatrix} e_1[0] & \cdots & e_n[0] \\ \vdots & & \vdots \\ e_1[l-1] & & e_n[l-1] \end{bmatrix}} = \begin{bmatrix} \vec{x}[1]^T \\ \vec{x}[2]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix} = [\vec{x}_1 \cdots \vec{x}_n] \ \star$$

Let $\begin{bmatrix} A^T \\ B^T \end{bmatrix} = [\vec{p}_1 \cdots \vec{p}_n]$, $\vec{p}_i$ being column vectors. $\to D[\vec{p}_1 \cdots \vec{p}_n] + [\vec{e}_1 \cdots \vec{e}_n]$

$\to [D\vec{p}_1 \ D\vec{p}_2 \cdots D\vec{p}_n] + [\vec{e}_1 \cdots \vec{e}_n] = [\vec{x}_1 \cdots \vec{x}_n] \to \underline{D\vec{p}_i + \vec{e}_i = \vec{x}_i}$

If D^TD is invertible, then $\hat{p}_i = (D^TD)^T D^{-1} \vec{x}_i$! $\to \begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix} = [\hat{p}_1 \cdots \hat{p}_n]$

$$\to \begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix} = [\hat{p}_1 \cdots \hat{p}_n] = (D^TD)^T D^{-1} [\vec{x}_1 \cdots \vec{x}_n] = \underline{(D^TD)^T D^{-1} \begin{bmatrix} \vec{x}[1]^T \\ \vdots \\ \vec{x}[l]^T \end{bmatrix}}$$

Stability

Scalar model, remove control input u: $x[i+1] = \lambda \vec{x}[i] + e[i]$

→ Does the sequence $\{x[0], x[1], \dots\}$ remain bounded?

- Take $\lambda = 2$ and ignore disturbance e: $x[i+1] = 2x[i]$.

→ $x[1] = 2x[0]$, $x[2] = 2x[1] = 4x[0]$, $x[3] = 8x[0]$

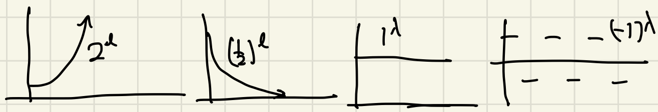
$x[l] = 2^l x[0]$ → blows up unless $x[0] = 0$.

Even with $x[0] = 10^{-9}$, $x[40] = 2^{40} x[0] \simeq 1000$ → not good

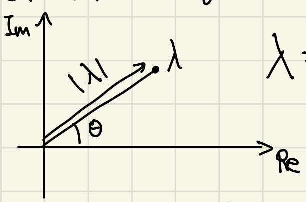
- Take $\lambda = 1/2$: $x[l] = \left(\frac{1}{2}\right)^l x[0]$ → bounded and $x[l] \to 0$ as $l \to \infty$.

For general λ, solution of $x[i+1] = \lambda x[i]$ is $x[l] = \lambda^l x[0]$.

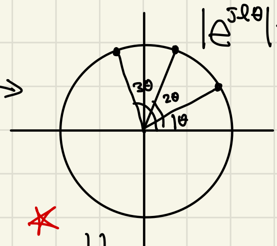
→ Bounded if $|\lambda| \leq 1$. *



For $\lambda \in \mathbb{C}$? When does λ^l remain bounded?



$\lambda = |\lambda| e^{j\theta} \to \lambda^l = |\lambda|^l \underline{e^{jl\theta}}$ →



$|e^{jl\theta}| = 1$

→ $|\lambda^l| = ||\lambda|^l e^{jl\theta}| = |\lambda|^l$

→ $|\lambda| \leq 1$ applies for boundness for $\lambda \in \mathbb{C}$ as well. *

Is $|\lambda| = 1$ really safe? → not really, when $e[i]$ is present.

$x[i+1] = x[i] + e[i] \to x[1] = x[0] + e[0]$, $x[2] = \overbrace{x[1]}^{x[0]+e[0]} + e[1]$

→ $x[l] = x[0] + \sum_{i=0}^{l-1} e[i]$ → Even a small constant e makes x unbounded!

Definition of Stability: A system is (bounded-input, bounded state) stable if state x is bounded for any initial condition and any bounded disturbance.* Unstable otherwise: when x is not bounded for some initial condition and some bounded disturbance.*

When is the system $x[i+1] = \lambda x[i] + e[i]$ stable?

$|\lambda| > 1$: unstable (zero input, nonzero initial condition → unbounded)

$|\lambda| = 1$: unstable (previous example, $x[\ell] = x[0] + \ell$)

$|\lambda| < 1$: stable (...?)

Claim: If $|\lambda| < 1$, then for any $x[0]$ and any bounded input e, the solutions of $x[i+1] = \lambda x[i] + e[i]$ remain bounded.

Proof: $x[1] = \lambda x[0] + e[0]$, $x[2] = \lambda x[1] + e[1] = \lambda(\lambda x[0] + e[0]) + e[1]$

$x[3] = \lambda x[2] + e[2] = \lambda(\lambda^2 x[0] + \lambda e[0] + e[1]) + e[2]$

$\rightarrow x[\ell] = \underbrace{\lambda^{\ell} x[0]}_{\substack{\text{bounded \&} \\ \text{converges to } 0 \\ \text{as } \ell \to \infty \\ (\text{b/c } |\lambda| < 1)}} + \underbrace{\sum_{k=0}^{\ell-1} \left(\lambda^k e[\ell-1-k]\right)}_{S}$ → is S also bounded when e is a bounded sequence?

– there is a number M s.t. $|e[i]| \leq M$ for all i.

$$\left|\sum_{k=0}^{\ell-1} \lambda^k e[\ell-1-k]\right| \leq \sum_{k=0}^{\ell-1} \left|\lambda^k e[\ell-1-k]\right| = \sum_{k=0}^{\ell-1} |\lambda|^k \left|e[\ell-1-k]\right| \leq \sum_{k=0}^{\ell-1} |\lambda|^k \cdot M = M \sum_{k=0}^{\ell-1} |\lambda|^k$$

$$\rightarrow \sum_{k=0}^{\ell-1} |\lambda|^k \leq \sum_{k=0}^{\infty} |\lambda|^k = \frac{1}{1-|\lambda|} \Rightarrow \left|\sum_{k=0}^{\ell-1} \lambda^k e[\ell-1-k]\right| \leq M \cdot \frac{1}{1-|\lambda|} \text{ (bounded)}$$

$x[l] = \lambda^l x[0] + S$, $\lambda^l x[0]$ and S are both bounded.
Therefore, $x[l]$ remains bounded. //

Vector Case: $\vec{x}[i+1] = A\vec{x}[i] + \vec{e}[i]$, $\vec{x} \in \mathbb{R}^N, \vec{e} \in \mathbb{R}^N, A = \mathbb{R}^{N\times N}$.

Solution (by recursion): $\vec{x}[l] = A^l \vec{x}[0] + \sum_{k=0}^{l-1} A^k \vec{e}[l-1-k]$

When does $\vec{x}[l]$ remain bounded? A is a matrix...

eigenvectors of A, lin. independent → diagonalizable

↳ Split into scalar equations: $\vec{y} := V^{-1}\vec{x}$ where $V = [\vec{v}_1 \cdots \vec{v}_n]$

$\vec{y}[i+1] = V^{-1}\vec{x}[i+1] = V^{-1}A\vec{x}[i] + V^{-1}\vec{e}[i] = \overline{V^{-1}AV}\vec{y}[i] + \overline{V^{-1}\vec{e}}[i]$

$= \Lambda \vec{y}[i] + \tilde{\vec{e}}[i]$ $\left(A[\vec{v}_1 \cdots \vec{v}_n] = [\lambda_1\vec{v}_1 \cdots \lambda_n \vec{v}_n] = [\vec{v}_1 \cdots \vec{v}_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\right)$

$\rightarrow y_k[i+1] = \lambda_k y_k[i] + \tilde{e}_k[i]$ → scalar equation, bounded when $|\lambda| < 1$.

→ if <u>ALL</u> eigenvalues of A is less than 1, $\vec{y}$, and thus $\vec{x}$ ($V\vec{y}$), is bounded. //

What if A is not diagonalizable? → Can still bring it to in an upper triangular form. (believe me... for now)

$$\vec{y}[i+1] = \begin{bmatrix}\lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & \lambda_n\end{bmatrix}\vec{y}[i] + \overbrace{V^{-1}\vec{e}}^{\text{some matrix}}[i] \quad (\star \text{ is some number})$$

$\rightarrow y_n[i+1] = \lambda_n y_n[i] + (V^{-1}\vec{e}[i])_n \rightarrow |\lambda_n| < 1 \Rightarrow y_n$ is bounded.

$\rightarrow y_{n-1}[i+1] = \lambda_{n-1}y_{n-1}[i] + \underbrace{\star \cdot y_n[i]}_{\text{bounded}} + \underbrace{(V^{-1}\vec{e}[i])_{n-1}}_{\text{bounded}} = \lambda_{n-1}y_{n-1}[i] + \underbrace{b[i]}_{\text{bounded}}$

$\rightarrow |\lambda_{n-1}| < 1 \Rightarrow y_{n-1}$ is bounded. $\Rightarrow$ Repeat for $(n:-1:1)$

$\Rightarrow \vec{x}[i]$ is stable if $|\lambda_k| < 1$ for all k (inside the complex unit circle)

Stability in continuous time systems: Same stability definition as x_d.

Different criteria for stability... $\frac{d}{dt}x(t) = \lambda x(t) + w(t)$ ← disturbance

$\rightarrow x(t) = \underbrace{e^{\lambda t}}\, x(0) + \underbrace{\int_0^t e^{\lambda(t-\tau)} w(\tau)\, d\tau} \rightarrow Re\{\lambda\} < 0 \Rightarrow e^{\lambda t} \rightarrow 0$ as $t \rightarrow \infty$

$\left(e^{\lambda_c t} = e^{\lambda_r t} \cdot e^{j\lambda_i t} \rightarrow |e^{\lambda_c t}| = |e^{\lambda_r t}|,\ |e^{j\lambda_i t}| = |\cos(\lambda_i t) + j\sin(\lambda_i t)| = 1\right)$

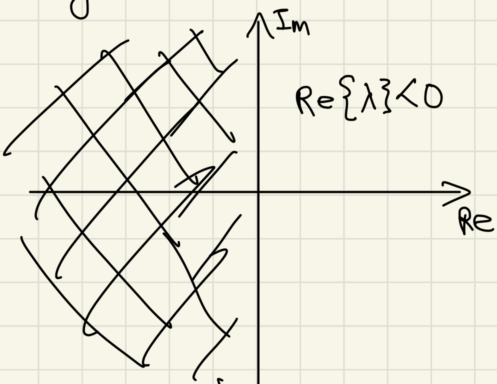
$\Rightarrow Re\{\lambda\} < 0 \rightarrow x(t)$ is stable. ($\int_0^t e^{\lambda(t-\tau)} w(\tau)\, d\tau$ is bounded?)

$Re\{\lambda\} = 0 \rightarrow x(t)$ is unstable ($x(t) = x(0) + \int_0^t w(\tau)\, d\tau$)

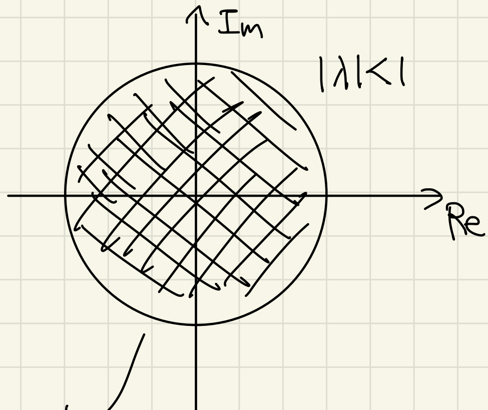
$Re\{\lambda\} > 0 \rightarrow x(t)$ is unstable ($x(t) = e^{\lambda t} x(0)$ even for $w(t) = 0$)

Vector case: $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{w}(t)$ Same arguments, if $Re\{\lambda_k\} < 0$ for all k of λ_k in A, $\vec{x}$ is stable.

Summary: continuous



discrete



λ_k must be inside to be stable.

Stabilization by Feedback

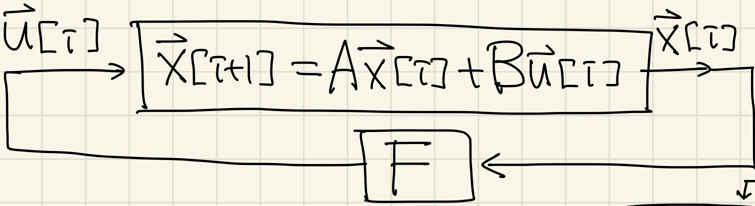
$$\vec{x}[i+1] = A\vec{x}[i] + B\overset{\text{control}}{\vec{u}[i]} + \overset{\text{disturbance}}{w[i]}$$

What if A has eigenvalue with $|\lambda| > 1$?

Can we achieve stability by designing $\vec{u}$?

$\rightarrow$ Try: $\vec{u}[i] = F\vec{x}[i]$ $\quad (\vec{u} \in \mathbb{R}^m, \vec{x} \in \mathbb{R}^n, F \in \mathbb{R}^{m\times n})$

if $m=1 \rightarrow F = \mathbb{R}^{1\times n} = [f_1 \ f_2 \cdots f_n] \rightarrow \vec{u}[i] = f_1 x_1[i] + \cdots + f_n x_n[i]$



"feedback"

Substitute $\vec{u}[i] = F\vec{x}[i] \rightarrow \vec{x}[i+1] = \overbrace{(A+BF)}^{A_{CL}}\vec{x}[i] + w[i]$ $\quad$ closed loop system

$\rightarrow$ Can we design F s.t. eigenvalues of A_{CL} are $|\lambda_{CL}| < 1$?

ex1) Scalar case: $x[i+1] = 2x[i] + u[i] \rightarrow$ unstable w/o feedback

$u[i] = f\cdot x[i] \rightarrow x[i+1] = (2+f)x[i] \rightarrow |2+f| < 1$

$\rightarrow -1 < 2+f < 1 \rightarrow \underbrace{-3 < f < -1}$ achieved stability

ex2) 2x2 case: $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\vec{u}[i] \rightarrow A_{CL} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[f_1 \ f_2] = \begin{bmatrix} 0 & 1 \\ 3+f_1 & 2+f_2 \end{bmatrix}$

$\det(\lambda I - A) = \lambda(\lambda-2) - 3 \rightarrow \lambda = 3 \text{ or } -1 \rightarrow$ unstable, need F to stabilize!

$\det(\lambda I - (A+BF)) = \det\left(\begin{bmatrix} \lambda & -1 \\ -3 & \lambda-2-f_2 \end{bmatrix}\right) = \lambda^2 - \underline{(2+f_2)}\lambda - \underline{(3+f_1)} \rightarrow \lambda = \lambda_1, \lambda_2$ (desired)

$\rightarrow (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - \underline{(\lambda_1+\lambda_2)} + \underline{\lambda_1\lambda_2} = 0 \Rightarrow \begin{cases} 2+f_2 = \lambda_1+\lambda_2 \\ -3-f_1 = \lambda_1\lambda_2 \end{cases} \rightarrow \begin{cases} f_1 = -3-\lambda_1\lambda_2 \\ f_2 = \lambda_1+\lambda_2-2 \end{cases}$

Does this always work? Not for any A, B.

ex3) $\vec{x}[i+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{u}[i] \rightarrow \lambda_A = 1$ or $2 \rightarrow$ unstable

$A_{CL} = A + BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix} \rightarrow \lambda_{A_{CL}} = (1+f_1)$ or $\underline{\underline{2}}$

$\rightarrow$ 2 can't be changed, unstable regardless of F

Controller Canonical Form: A special structure of A and B in which we can arbitrary assign eigenvalues of $A_{CL} = A + BF$ with the choice of F.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(Example 2 had this form, $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.)

$n=2, a_1=3, a_2=2$

Nice Properties of this form:

1) Char. poly. of A is transparent: $\det(\lambda I - A) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_1$

2) $A+BF$ has the same structure as A. $a_k \rightarrow a_k + f_k,\ k \in [1, n]$.

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ (a_1+f_1) & \cdots & & (a_n+f_n) \end{bmatrix}$$

$\rightarrow$ From 1 and 2, $\det(\lambda I - A_{CL}) = \lambda^n - (a_n+f_n)\lambda^{n-1} - (a_2+f_2)\lambda^{n-2} \cdots - (a_1+f_1)$.

Suppose we want $A_{CL} = A + BF$ to have $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$. Then, the char. poly. of A_{CL} should be: $(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$, which is guaranteed for some F.

$(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = \lambda^n - (\sum_{i=1}^n \lambda_i)\lambda^{n-1} \cdots + ((-1)^n \prod_{i=1}^n \lambda_i)$

$\rightarrow a_1 + f_1 = -(-1)^n \prod_{i=1}^n \lambda_i, \ \ldots, \ a_n + f_n = \sum_{i=1}^n \lambda_i \rightarrow \{f_1, \ldots, f_n\}$ has a closed form!

Can we bring A,B to canonical form by a change of variables?

$\vec{y} = T\vec{x}$, $T \in \mathbb{R}^{n\times n}$, invertible, TBD

$\vec{y}[i+1] = T\vec{x}[i+1] = T(A\vec{x}[i] + Bu[i]) = TA\vec{x}[i] + TBu[i]$

$= \underbrace{TAT^{-1}}_{A'}\vec{y}[i] + \underbrace{TB}_{B'}u[i] \rightarrow A', B'$ should be in canonical form.

$TAT^{-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & \ddots & \\ a_1 & \cdots & & a_n \end{bmatrix}$, $TB = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \rightarrow$ Can we find such T?

Claim: Yes, if $[A^{n-1}B, A^{n-2}B, \cdots, AB, B]$ $(\in \mathbb{R}^{n\times n})$ is invertible.

ex3) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow [AB, B] = \left[\begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array}\right] \rightarrow$ not invertible

When $[A^{n-1}B, \cdots, AB, B]$ is invertible, feedback design is easy

in $\vec{y}$ coordinates: $\vec{y}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & & \ddots & 1 \\ a_1 & \cdots & & a_n \end{bmatrix}}_{A_y}\vec{y}[i] + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}}_{B_y}u[i]$, $u[i] = F_y \cdot y[i]$

Can assign eigenvalues of $A_{CLY} = A_y + B_y F_y$ b/c A_y, B_y is in canonical form.

$\rightarrow u[i] = F_y \vec{y}[i] = \underbrace{F_y T}\vec{x}[i] \rightarrow F = F_y T$

$\Rightarrow \vec{x}[i+1] = \underbrace{(A+BF)}_{A_{CL}}\vec{x}[i]$

* evalues of $A_{CL} = A + BF$ are same as $A_{CLY} = A_y + B_y F_y$ (which were designed by F_y)

$(A + B\underset{F_yT}{\underset{\parallel}{F}})\vec{v} = \lambda\vec{v}$, $(\underset{TAT^{-1}}{\underset{\downarrow}{A_y}} + \underset{TB}{\underset{\downarrow}{B_y}}\underset{F_y}{\underset{\downarrow}{F_y}})\vec{v_y} = \lambda\vec{v_y}$

$(TAT^{-1} \ TBF_y)\underline{(T\vec{v})} = \lambda\underline{(T\vec{v})}$

Proof of Claim: Let q^T be the top row of $[A^{n-1}B \ A^{n-2}B \cdots B]^{-1}$.

$\rightarrow \begin{bmatrix} q^T \\ \vdots \\ \end{bmatrix} [A^{n-1}B \ \ A^{n-2}B \cdots \ B] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$

$\rightarrow q^T A^{n-1}B = 1,\ q^T A^{n-2}B = 0 \cdots q^T AB = 0,\ q^T B = 0.$

Take $T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$. $TB = \begin{bmatrix} q^T B \\ q^T AB \\ \vdots \\ q^T A^{n-1} B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. $TA = \begin{bmatrix} q^T A \\ q^T A^2 \\ \vdots \\ q^T A^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 1 \\ * & * & \cdots & & * \end{bmatrix} \underbrace{\begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}}_{T}$

$\rightarrow TAT^{-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \\ * & \cdots & & * \end{bmatrix}$

* the condition of the claim is sufficient to be able to assign e-values of $A+BF$ by choice of F. No need to change variables.

ex1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $n=2 \rightarrow$ is $[AB \ \ B]$ lin. ind.?

$[AB \ \ B] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \rightarrow$ Yes. $A+BF = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \ f_2] = \begin{bmatrix} 1 & 1 \\ f_1 & (2+f_2) \end{bmatrix}$

* evalues of A are $\{1,2\}$. for A_{CL}, $\det\left(\begin{bmatrix} 1-\lambda & 1 \\ f_1 & 2+f_2-\lambda \end{bmatrix}\right) \rightarrow \lambda^2 - (3+f_2)\lambda + 2 + f_2 - f_1 = 0$

assume we want $\lambda_{1,2} = \{0,0\}$. $\rightarrow \lambda^2 = 0 \rightarrow \begin{cases} +3+f_2 = 0 \\ 2+f_2-f_1 = 0 \end{cases} \rightarrow (f_1, f_2) = (-1, -3)$

ex2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$ lin. dep. on B.

$A+BF = \begin{bmatrix} 1+f_1 & 1+f_2 \\ 0 & 2 \end{bmatrix} \rightarrow 2$ is still unchanged. $\rightarrow$ always unstable

Controllability

Recall: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$ (assume single input)

$i=0$: $\vec{x}[1] = A\vec{x}[0] + Bu[0]$, $\vec{x}[2] = A(A\vec{x}[0] + Bu[0]) + Bu[1]$,

$\vec{x}[3] = A\vec{x}[2] + Bu[2] = A^3\vec{x}[0] + A^2Bu[0] + ABu[1] + u[2]$

$$\rightarrow \vec{x}[l] - A^l x[0] = \begin{bmatrix} A^{l-1}B & A^{l-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[l-1] \end{bmatrix}$$

Can we find an input sequence $u[0] \ldots u[l-1]$ that brings state $\vec{x}$ from $\vec{x}[0]$ to target $\vec{x}_{target}$ at time l?

$\rightarrow$ Yes, if $\vec{x}_{target} - A^l x[0]$ lies in $\mathrm{Col}\{C_l := [A^{l-1}B \cdots AB\ B]\}$.

"Controllability means the ability to reach any target $\vec{x}_{target}$ from any $\vec{x}[0]$."

Definition: A system is controllable if given any target state $\vec{x}_{target}$ and $\vec{x}[0]$, we can find a time l and input sequence $u[0], \ldots, u[l-1]$ s.t. $\vec{x}[l] = \vec{x}_{target}$.

Test for controllability: If C_l has <u>n linearly dependent columns</u> for some l, then $\mathrm{Col}\{C_l\} = \mathbb{R}^n$, which means we can make $C_l \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}$ anything we want by choosing $u[i]$. Specifically, assign $C_l \begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix} = \vec{x}_{target} - A^l \vec{x}[0]$. $\rightarrow \vec{x}[l] = \vec{x}_{target}$.

ex1) $B=\begin{bmatrix}0\\1\end{bmatrix}$ $AB=\begin{bmatrix}1\\2\end{bmatrix}$, $C_1=B$, dim$=1$, $C_2=\begin{bmatrix}1&0\\2&1\end{bmatrix}$, dim$=2=n$ ✓

ex2) $B=\begin{bmatrix}1\\0\end{bmatrix}$ $A=\begin{bmatrix}1&1\\0&2\end{bmatrix}$ $\rightarrow C_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $C_3=\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}$ $\rightarrow$ dim always 1!

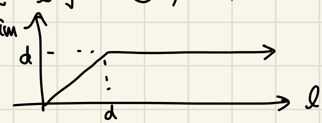
Lemma: If A^lB is linearly dependent on $\{A^{l-1}B, \dots, AB, B\}$, then $A^{l+1}B$ is also linearly dependent on them.

Proof: $A^lB = \alpha_{l-1}A^{l-1}B + \alpha_{l-2}A^{l-2}B + \cdots + \alpha_1 AB + \alpha_0 B$ for some α_i.

$$A^{l+1}B = A(A^lB) = A(\alpha_{l-1}A^{l-1}B + \cdots + \alpha_1 AB + \alpha_0 B)$$
$$= \alpha_{l-1}\underbrace{A^lB} + \alpha_{l-2}A^{l-1}B + \cdots + \alpha_1 A^2B + \alpha_0 AB$$
$$= \beta_{l-1}A^{l-1}B + \beta_{l-2}A^{l-2}B + \cdots + \beta_1 AB + \beta_0 B \quad //$$

$* \; C_{l+1} = [A^lB \; \overbrace{\boxed{A^{l-1}B \cdots AB \;\; B}}^{C_l}]$

Lemma implies that if Col$\{C_{l+1}\}$ = Col$\{C_l\} = d$, then Col$\{C_{l+i}\}$ is also d for $i \geq 0$.



If ① $d<n \rightarrow$ uncontrollable, ② $d=n \rightarrow$ controllable

$\Rightarrow$ Check C_n. If it is full rank (Col$\{C_n\}=n$), controllable. *

Condition for feedback design = Condition for controllability

$= \text{rank}([A^{n-1}B \;\; A^{n-2}B \;\; \cdots \;\; AB \;\; B]) = n$

ex1) $A=\begin{bmatrix}1&1\\0&2\end{bmatrix}, B=\begin{bmatrix}1\\0\end{bmatrix}$

$n=2 \rightarrow C_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$

$\rightarrow$ uncontrollable

$x_1[i+1]=x_1[i]+x_2[i]+u[i]$
$x_2[i+1]=2\cdot x_2[i]$

$\rightarrow x_2[l]=2^l x_2[0]$

① eigenvalue of 2 remains regardless of feedback

② can't take x_2 component to whereever we want

ex2) $A=\begin{bmatrix}1&1\\0&2\end{bmatrix}, B=\begin{bmatrix}0\\1\end{bmatrix}$

$n=2 \rightarrow C_2=\begin{bmatrix}1&0\\2&1\end{bmatrix}$

$\rightarrow$ controllable

$x_1[i+1]=x_1[i]+x_2[i]$
$x_2[i+1]=2\cdot x_2[i]+u[i]$

u doesn't appear in x_1 but can influence x_1 through x_2

Orthonormality & Gram-Schmidt

Orthonormality: Column vectors $\vec{q}_1 \dots \vec{q}_n$ are orthonormal if

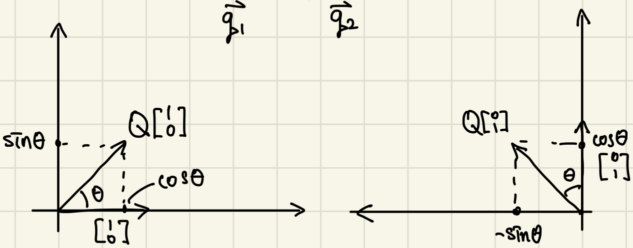
$$q_i^T q_j = \begin{cases} 0 & (i \neq j) \rightarrow \text{orthogonality} \\ 1 & (i = j) \rightarrow \text{normality} \end{cases}.$$

A matrix $Q = [\vec{q}_1 \cdots \vec{q}_k]$ with orthonormal columns satisfies:

$$\underline{Q^TQ} = \begin{bmatrix} - \vec{q}_1^{\,T} - \\ \vdots \\ - \vec{q}_k^{\,T} - \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_k \end{bmatrix} = \begin{bmatrix} \vec{q}_1^{\,T}\vec{q}_1 & \cdots & \\ \vdots & \ddots & \\ & & \vec{q}_k^{\,T}\vec{q}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} = \underline{I_{k \times k}}$$

If Q is square, $Q^TQ = I \Leftrightarrow Q^T = Q^{-1} \Leftrightarrow QQ^T = I$.
(Q is called orthogonal, in this case.)

ex) $Q = \begin{bmatrix} \underbrace{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}}_{\vec{q}_1} & \underbrace{\begin{matrix}-\sin\theta \\ \cos\theta\end{matrix}}_{\vec{q}_2} \end{bmatrix} \rightarrow \vec{q}_1^{\,T}\vec{q}_2 = 0,\ \vec{q}_i^{\,T}\vec{q}_i = \cos^2\theta + \sin^2\theta = 1.$



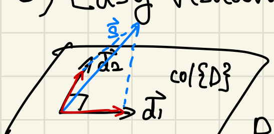
Useful features of matrices of orthonormal columns:

1) $\underbrace{\|Q\vec{x}\|} = \underbrace{\|\vec{x}\|}$ (preserves magnitude)

$\left(\sqrt{(Q\vec{x})^T(Q\vec{x})} = \sqrt{\vec{x}^T \cancel{Q^TQ} \vec{x}} = \sqrt{\vec{x}^T\vec{x}}\right)$

2) $(Q\vec{x})^T(Q\vec{y}) = \vec{x}^TQ^TQ\vec{y} = \vec{x}^T\vec{y}$ (preserves dot product)

3) Easy visualization of columnspace: for $D = [\vec{d}_1\ \vec{d}_2]$, if orthonormal:



projections onto columnspace is trivial.

$$\text{Proj}_{(col\{D\})}\vec{S} = (\vec{d}_1^{\,T}\vec{S})\vec{d}_1 + (\vec{d}_2^{\,T}\vec{S})\vec{d}_2.$$

Recall Least Squares: $\vec{s} \simeq D\vec{p} \rightarrow \hat{\vec{p}} = (D^TD)^{-1}D^T\vec{s}$.

What if D had orthonormal columns? $\hat{\vec{p}} = D^T\vec{s}$! (no inversion)

Gram-Schmidt: Even if columns of D are not orthonormal, we can construct an orthonormal basis for the columnspace close to the original column in the sense that ...

i-th column $\vec{d_i}$ is a combination of $\vec{q_1}, \dots, \vec{q_i}$, i.e. d_i can be constructed by $\vec{q_1}$, $\vec{d_2}$ by $\vec{q_1}$ & $\vec{q_2}$, and so on.

There fore, $[\vec{d_1} \cdots \vec{d_k}] = \underbrace{[\vec{q_1} \cdots \vec{q_k}]}_{Q} \begin{bmatrix} \star & \star & \cdots & \star \\ 0 & \star & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & \star \end{bmatrix} \rightarrow R$ (upper triangular)

(a.k.a. Q-R factorization)

Back to least squares: $\vec{s} = D\vec{p} + \vec{e}$, pick $\hat{\vec{p}}$ s.t. $\vec{e} \perp \mathrm{col}\{D\}$

$\rightarrow D^T(\vec{s} - D\hat{\vec{p}}) = 0 \rightarrow D^T\vec{s} = D^TD\hat{\vec{p}}$.

Instead of inversion, write $D = QR$. $\rightarrow (QR)^T\vec{s} = (QR)^T(QR)\hat{\vec{p}}$.

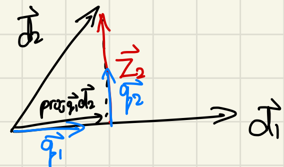
$\rightarrow R^TQ^T\vec{s} = R^T\cancelto{I}{Q^TQ}R\hat{\vec{p}} \rightarrow R^TQ^T\vec{s} = R^TR\hat{\vec{p}} \rightarrow R\hat{\vec{p}} = Q^T\vec{s}$.

$\begin{bmatrix} \star & \cdots & \star \\ 0 & \ddots & \vdots \\ 0 & \cdots 0 & \star \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_k \end{bmatrix} = Q^T\vec{s} \rightarrow$ last row gives $\star \hat{p}_k = (Q^T\vec{s})_k \Rightarrow \hat{p}_k = (Q^T\vec{s})_k \cdot \frac{1}{\star}$

second to last row gives $\star p_{k-1} + \star p_k = (Q^T\vec{s})_{k-1} \rightarrow p_{k-1}$ solved linearly

$\Rightarrow$ can solve by back substitution! (faster than $(D^TD)^{-1}$)

Gram-Schmidt Algorithm:



① $\vec{q}_1 = \frac{1}{\|\vec{d}_1\|}\vec{d}_1$

② $\vec{Z}_2 = \vec{d}_2 - \text{proj}_{(\vec{q}_1)}\vec{d}_2 = \vec{d}_2 - (\vec{d}_2^{\,T}\vec{q}_1)\vec{q}_1, \quad \vec{q}_2 = \frac{1}{\|\vec{Z}_2\|}\vec{Z}_2$

③ $\vec{Z}_3 = \vec{d}_3 - \text{proj}_{(\vec{q}_1)}\vec{d}_3 - \text{proj}_{(\vec{q}_2)}\vec{d}_3, \quad \vec{q}_3 = \frac{1}{\|\vec{Z}_3\|}\vec{Z}_3$

⋮

ⓚ $\vec{Z}_k = \vec{d}_k - \sum_{i=1}^{k-1}\text{proj}_{(\vec{q}_i)}d_k = \vec{d}_k - \sum_{i=1}^{k-1}(\vec{d}_k^{\,T}\vec{q}_i)\vec{q}_i, \quad \vec{q}_k = \frac{1}{\|\vec{Z}_k\|}\vec{Z}_k$

$*\ \vec{Z}_k^{\,T}\vec{q}_i = 0$ for $i<k \rightarrow \vec{q}_k^{\,T}\vec{q}_i = 0$ for $i<k$

$\hookrightarrow \vec{d}_k^{\,T}\vec{q}_i - \sum_{j=1}^{k-1}(\vec{d}_k^{\,T}\vec{q}_j)\underbrace{\vec{q}_j^{\,T}\vec{q}_i}_{\begin{cases}0 & (i\neq j)\\ 1 & (i=j)\end{cases}} = \vec{d}_k^{\,T}\vec{q}_i - \vec{d}_k^{\,T}\vec{q}_i = 0$

$\Rightarrow$ Orthonormal

Upper Triangularization

Recall: Diagonalization, $n \times n$ matrix with <u>n lin. indep. evectors</u>

$$A \underbrace{[\vec{v}_1 \cdots \vec{v}_n]}_{V} = [\lambda_1 \vec{v}_1 \cdots \lambda_n \vec{v}_n] = \underbrace{[\vec{v}_1 \cdots \vec{v}_n]}_{V} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\Rightarrow V^{-1} A V = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Today: We can <u>always</u> upper-triangularize even if we can't diagonalize.

→ Upper triangular form has some benefits of a diagonal matrix.

① E.values are the diagonal entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots \\ 0 & a_{22} & \cdots & \cdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \qquad \lambda I - A = \begin{bmatrix} (\lambda - a_{11}) & - & - & - & - \\ 0 & (\lambda - a_{22}) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & (\lambda - a_{nn}) \end{bmatrix}$$

if $\lambda = a_{11}$ → zero column for first column → rank drops → a_{11} is an eval.

if $\lambda = a_{nn}$ → zero row for last row → rank drops → a_{nn} is an eval.

if $\lambda = a_{ii}$ → $\lambda I - A = \underbrace{\begin{bmatrix} (a_{ii} - a_{11}) & & & \\ & \ddots & & \\ & & 0 & \\ & & & \ddots \; (a_{ii} - a_{nn}) \end{bmatrix}}$ → i columns, only top $(i-1)$ nonzero entries → span at most $(i-1)$ dim.
$(1 < i < n)$

i-th col, row

↳ linearly dependence in first i columns → $\lambda I - A$ is lin. dep.

→ not full rank → a_{ii} is an eval.

② Solution of a VDE or DE can be broken down to scalar eq.s.

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} a_{11} & - & - & - \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & - & - 0 & a_{nn} \end{bmatrix}\vec{x}(t) + B\vec{u}(t) \rightarrow \frac{d}{dt}x_n(t) = a_{nn}x_n(t) + \underbrace{\left(B\vec{u}(t)\right)_n}$$

$$\rightarrow \underbrace{x_n(t)} = e^{a_{nn}(t-t_0)} + x_n(t_0) + \int_{t_0}^{t} \_\_\_\_$$

$$\rightarrow \frac{d}{dt}x_{n-1}(t) = a_{(n-1)(n-1)}x_{n-1}(t) + \underbrace{a_{(n-1)n}x_n(t) + \left(B\vec{u}(t)\right)_{n-1}}_{\text{known function} \rightarrow \text{treat as input}}$$

$$\rightarrow x_{n-1}(t) = e^{a_{(n-1)(n-1)}(t-t_0)}x_{n-1}(t_0) + \int_{t_0}^{t} \_\_\_\_$$

Likewise, backsubstitute until first row.

ex) longitudinal motion of a car described by:

$p(t)$:= position, $V(t)$:= velocity, M := mass, R := radius of tire

$$\frac{d}{dt}p(t) = V(t),\quad M\cdot\frac{d}{dt}V(t) = \frac{1}{R}\cdot u(t) \rightarrow u(t) := \text{torque}$$

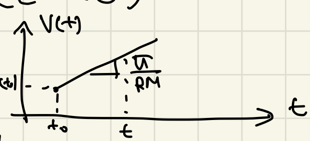
$$\rightarrow \frac{d}{dt}\underbrace{\begin{bmatrix} p(t) \\ V(t) \end{bmatrix}}_{\vec{x}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_c}\begin{bmatrix} p(t) \\ V(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ \frac{1}{MR} \end{bmatrix}}_{B_c}u(t) \rightarrow \text{evalue of } A_c: \{0,0\}, \text{evector}: \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

no two lin. ind. evectors → non diagonalizable

Suppose $u(t) = \bar{u}$ = constant → Solution of DE?

$$\frac{d}{dt}V(t) = \frac{1}{RM}\bar{u} \rightarrow V(t) = V(t_0) + \frac{\bar{u}}{RM}(t - t_0)$$

$$\frac{d}{dt}p(t) = V(t) = V(t_0) + \frac{\bar{u}}{RM}(t - t_0)$$



$$\rightarrow p(t) = p(t_0) + V(t_0)(t - t_0) + \frac{1}{2}\frac{\bar{u}}{RM}(t - t_0)^2$$

→ can find discrete-time model if we set:

$$t_0 = i\Delta,\ t = (i+1)\Delta,\ u(t) = u_d[i] \rightarrow p_d[i+1] = \underline{p_d[i] + \Delta V_d[i] + \frac{\Delta^2}{2RM}u_d[i]}$$

$$\underline{V[i+1] = V[i] + \frac{\Delta}{RM}u_d[i]}$$

Now: Prove that any square matrix can be upper-triangularized.

Theorem: For any $n \times n$ matrix A, we can find an orthogonal matrix U s.t. $\underline{U^{-1}AU}$ is upper triangular $\rightarrow$ $\underline{U^TAU}$ as well

Thus, if $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$, then $\vec{y} = U^T\vec{x}$,

$$\frac{d}{dt}\vec{y}(t) = \underbrace{U^TAU}_{\text{upper-triangular}}\vec{y}(t) + U^TB\vec{u}(t)$$

Proof: By Induction. $S_n :=$ Theorem statement

- Show S_1 is true, Show S_{k+1} is true if S_k is true

S_1: True b/c scalars are upper-triangular

Assume S_k is true $\rightarrow$ $\underline{\text{Any } k \times k \text{ matrix has an orthogonal } U \text{ for upper-tri.}}$ I.H.

① Let $A :=$ arbitrary $(k+1) \times (k+1)$ matrix and $\lambda_1, \vec{q}_1$ be an eigenvalue-vector pair that is real (easy to adapt to complex). Assume WLOG $\|\vec{q}_1\| = 1$.

② Choose an orthonormal basis for $\mathbb{R}^{(k+1)}$ that includes $\vec{q}_1$: $\{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_{(k+1)}\}$

↳ How? Pick k vectors in $\mathbb{R}^{(k+1)}$ s.t. when combined with $\vec{q}_1$ form a basis for $\mathbb{R}^{k+1}$.

Then, use Gram-Schmidt $\rightarrow$ $\vec{q}_1$ remains as an orthonormal vector.

③ Then $Q = [\vec{q}_1, \dots, \vec{q}_{(k+1)}]$ is an orthogonal matrix.

$$\rightarrow AQ = \begin{bmatrix} A\vec{q}_1 & A\vec{q}_2 & \cdots & A\vec{q}_{(k+1)} \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{q}_1 & A_2\vec{q}_2 & \cdots & A\vec{q}_{(k+1)} \end{bmatrix}$$

$$\rightarrow Q^TAQ = \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_{k+1}^{\,T} \end{bmatrix} \begin{bmatrix} \lambda_1\vec{q}_1 & A_2\vec{q}_2 & \cdots & A\vec{q}_{(k+1)} \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{q}_1^{\,T}\vec{q}_1 & * & \cdots & * \\ \lambda_1\vec{q}_2^{\,T}\vec{q}_1 & & \ddots & \vdots \\ \vdots & & & \\ \lambda_1\vec{q}_{k+1}^{\,T}\vec{q}_1 & * & \cdots & * \end{bmatrix} = \begin{bmatrix} \lambda_1 & \boxed{* \; \cdots \; *} \\ 0 & \boxed{\begin{matrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{matrix}} \\ \vdots & \\ 0 & \end{bmatrix}$$

$\vec{p}^T$ $\quad A_0\ (k \times k)$

Summary: $Q^TAQ = \begin{bmatrix} \lambda_1 & - \vec{p}^T - \\ 0 & \\ \vdots & [\ A_0\ ] \\ 0 & \end{bmatrix}$, $A_0 \in \mathbb{R}^{k \times k}$

By induction hypothesis, A_0 is upper-triangularizable

$\rightarrow \exists$ orthogonal U_0 s.t. $U_0^T A_0 U_0$ is upper-triangular

Define $U := Q \begin{bmatrix} 1 & \\ & [U_0] \end{bmatrix}$, which is orthogonal ($U^T = \begin{bmatrix} 1 & \\ & [U_0^T] \end{bmatrix} \rightarrow U^TU = I$)

$$U^TAU = \begin{bmatrix} 1 & \\ & [U_0^T] \end{bmatrix} Q^TAQ \begin{bmatrix} 1 & \\ & [U_0] \end{bmatrix} = \begin{bmatrix} 1 & \\ & [U_0^T] \end{bmatrix} \begin{bmatrix} \lambda_1 & - \vec{p}^T - \\ 0 & \\ \vdots & [\ A_0\ ] \\ 0 & \end{bmatrix} \begin{bmatrix} 1 & \\ & [U_0] \end{bmatrix}$$

$= \begin{bmatrix} \lambda_1 & - \vec{p}^T U_0 - \\ 0 & \\ \vdots & [\ U_0^T A_0 U_0\ ] \\ 0 & \end{bmatrix} \rightarrow$ since $U_0^T A_0 U_0$ is upper-tri., U^TAU is upper-tri. //

Notes:

① A and T have the same eigenvalues. If $(\lambda, \vec{v})$ is an e-value & vector for T, then $(\lambda, U\vec{v})$ is an e-value & vector for A.

$$\rightarrow \underline{A(U\vec{v})} = UTU^T(U\vec{v}) = UT\vec{v} = U\lambda\vec{v} = \underline{\lambda(U\vec{v})}$$

② Diagonal entries of T are its e-values

$\Rightarrow$ Once matrix A is upper-triangularized, its e-values appear in the diagonal entries of T.

$$U^TAU = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

③ Induction proof can be turned into a recursive algorithm

Given $A \in \mathbb{R}^{(k+1)\times(k+1)}$ with real e-values:

Define Triangularize(A) :=

- pick a evalue & vector pair $(\lambda_1, \vec{q}_1)$, $\vec{q}_1$ is normal.
- Gram-Schmidt for $\mathbb{R}^{(k+1)}$ using $\vec{q}_1 \to \{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_{k+1}\}$ (orthonormal)

  $Q = [\vec{q}_1 \; \vec{q}_2 \cdots \vec{q}_{k+1}]$. Then $Q^T A Q = \begin{bmatrix} \lambda_1 & \text{---}\ \vec{p}^T \ \text{---} \\ \vec{0} & [\ A_0\ ] \end{bmatrix}$.
- Return Q, A_0

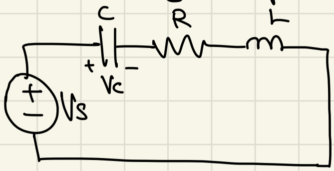
$(Q, A_0) :=$ Triangularize(A)

$U := Q$

while (size$(A_0) > 1$):

- $(Q, A_0) :=$ Triangularize(A_0)
- $U := U \cdot \begin{bmatrix} 1 & \text{---}\ \vec{0}^T \ \text{---} \\ \vec{0} & [\ Q\ ] \end{bmatrix}$

ex) Critically damped RLC circuit



In HW5: $\vec{x}(t) = V_c(t)$, $\vec{x}_2(t) = \frac{d}{dt}V_c(t)$

$$\rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad \lambda_{1,2} = -\frac{R}{2L} \mp \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}.$$

Suppose $\frac{R^2}{L^2} = \frac{4}{LC}$. $\rightarrow \lambda_1 = \lambda_2 = -\frac{R}{2L}$, $-\frac{1}{LC} = -\frac{R^2}{4L^2} \rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{R^2}{4L^2} & -\frac{R}{L} \end{bmatrix}$.

$$\lambda I - A = \begin{bmatrix} -\frac{R}{2L} & 0 \\ 0 & -R/2L \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{R^2}{4L^2} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} -\frac{R}{2L} & -1 \\ \frac{R^2}{4L^2} & \frac{R}{2L} \end{bmatrix}. \quad null(\lambda I - A) = \begin{bmatrix} 1 \\ -\frac{R}{2L} \end{bmatrix}\alpha, \ \alpha \neq 0.$$

Can't find 2 lin. ind. e-vectors → not diagonalizable!

→ Upper triangularize: $U^T A U = \begin{bmatrix} \lambda & \star \\ 0 & \lambda \end{bmatrix}$, $\lambda = \frac{-R}{2L}$ for some orthogonal U.

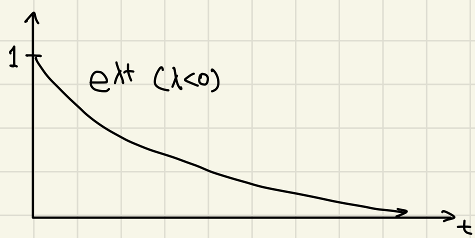
Solution of diff. eq.: $V_s = 0$ (for simplicity), $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \rightarrow \frac{d}{dt}\vec{y}(t) = U^T A U \vec{y}(t)$

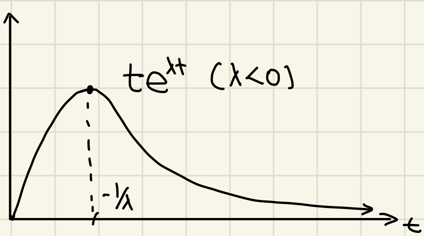
$$\rightarrow \begin{cases} \frac{d}{dt}y_1(t) = \lambda y_1(t) + \star y_2(t) \rightarrow \frac{d}{dt}y_1(t) = \lambda y_1(t) + \underbrace{\star y_2(0) e^{\lambda t}}_{u(t)} \\ \frac{d}{dt}y_2(t) = \lambda y_2(t) \rightarrow y_2(t) = e^{\lambda t} \cdot y_2(0) \end{cases}$$

$$\rightarrow y_1(t) = y_1(0)e^{\lambda t} + \int_0^t \underbrace{e^{\lambda(t-\tau)}(\star y_2(0) e^{\lambda\tau})}_{e^{\lambda t}\star y_2(0)} d\tau$$

signature of repeated evalues

$$\rightarrow y_1(t) = y_1(0)e^{\lambda t} + te^{\lambda t}\star y_2(0), \quad y_2(t) = y_2(0)e^{\lambda t}$$





Similarly, $t^2 e^{\lambda t}$ would appear in the solution to:

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} \lambda & \star & \star \\ 0 & \lambda & \star \\ 0 & 0 & \lambda \end{bmatrix}\vec{y}(t) \quad \text{(3 repeated evalues)}$$

$U^TAU = T \Rightarrow A = UTU^T$, U: orthogonal, T: upper-triangular

↳ Schur Decomposition

Spectral Theorem: For a diagonalizable matrix A, we can find V s.t. $V^{-1}AV = \underline{\begin{bmatrix} \lambda_1 & 0 & - & 0 \\ 0 & \ddots & \ddots & | \\ | & \ddots & \ddots & 0 \\ 0 & - & 0 & \lambda_n \end{bmatrix}}$. V here is not necessarily orthogonal.

If we instead upper-triangularize, we find <u>orthogonal</u> U s.t.

$U^{-1}AU = \underset{\sim\sim\sim}{U^TAU}$ is upper triangular $\begin{bmatrix} \lambda_1 & * & - & * \\ 0 & \ddots & \ddots & | \\ | & \ddots & \ddots & * \\ 0 & - & 0 & \lambda_n \end{bmatrix}$.

For <u>symmetric matrices ($A = A^T$)</u>, we get both.

$V^{-1}AV = \underset{\sim\sim\sim}{V^TAV} = \underline{\begin{bmatrix} \lambda_1 & 0 & - & 0 \\ 0 & \ddots & \ddots & | \\ | & \ddots & \ddots & 0 \\ 0 & - & 0 & \lambda_n \end{bmatrix}}$

- Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then:
  1. E-values of A are real.
  2. A is diagonalizable.
  3. E-vectors of A are pairwise orthogonal $\Rightarrow$ choose them to be length 1, then they constitute an orthonormal basis $\rightarrow V = [\vec{v}_1 \cdots \vec{v}_n]$ is orthogonal.

     $\Rightarrow V^{-1}AV = V^TAV = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$

Proof:

① Choose one $(\lambda, \vec{v})$ pair. $A\vec{v} = \lambda\vec{v}$. $\lambda = a + bj$, show $b = 0$, i.e. $\lambda = \bar{\lambda}$.

Take complex conjugates onboth sides: $\overline{(A\vec{v})} = \overline{(\lambda\vec{v})} \rightarrow \underbrace{\bar{A}}\,\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$.

A is real $\rightarrow A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$. Transpose: $\bar{\vec{v}}^T A^T = \bar{\vec{v}}^T \bar{\lambda}$. $\rightarrow \bar{\vec{v}}^T A = \bar{\vec{v}}^T \bar{\lambda}$

Multiply both sides by $\vec{v}$: $\bar{\vec{v}}^T A \vec{v} = \underbrace{\bar{\lambda}(\bar{\vec{v}}^T \vec{v})}$.

Multiply original by $\bar{\vec{v}}^T$: $\bar{\vec{v}}^T A \vec{v} = \underbrace{\lambda(\bar{\vec{v}}^T \vec{v})}$

$\rightarrow \bar{\lambda}(\bar{\vec{v}}^T \vec{v}) = \lambda(\bar{\vec{v}}^T \vec{v}) \Rightarrow \underline{\lambda = \bar{\lambda}}$ $\left(\bar{\vec{v}}^T \vec{v} = \sum_{i=1}^{n} \|v_i\| \neq 0\right)$ //

$\rightarrow$ E-vectors must also be real ($\lambda I - A = 0 \rightarrow \vec{v}$ can't be complex)

② Apply Schur Decomposition:

$$U^T A U = T \rightarrow T^T = U^T A^T (U^T)^T = U^T A U.$$

$\rightarrow$ <u>T is also symmetric.</u> <u>$T = T^T$</u> implies T is diagonal.

③ $U^T A U = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \rightarrow \cancel{UU^T} A U = U \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$

$$A[\vec{u}_1 \cdots \vec{u}_n] = [\vec{u}_1 \cdots \vec{u}_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} = \lambda_1\vec{u}_1 + \cdots + \lambda_n\vec{u}_n$$

$\rightarrow A\vec{u}_i = \lambda_i\vec{u}_i \rightarrow$ columns of orthogonal matrix U obtained from upper-triangularization are orthonormal basis for $\mathbb{R}^n$.

SVD

Once we learn SVD, we will be able to:

1) Perform "Principle Component Analysis" (PCA), application of SVD in statistics to find informative directions in a dataset
2) Find "minimum norm (energy)" solutions for $\underbrace{C}\vec{\omega} = \underbrace{\vec{z}}_{\text{given}}$

   where C is a wide matrix $\rightarrow$ nontrivial nullspace

   if a solution $\vec{\omega}_0$ exists, then there are infinitely many others;

   $\vec{\omega}_0 + \vec{n}$, $n \in \text{null}(C)$ is another solution. One way to select $\vec{\omega}$ is to pick one with the least norm, $\|\vec{\omega}\|$.

Why might we want to minimize the norm?

Consider a controllable system: $\vec{x}[i+1] = A\vec{x}[i] + Bu[i]$.

Suppose we want to reach $\vec{x}_{target}$ at timestamp l from $\vec{x}[0]$.

Then $u[0] \ldots u[l-1]$ must be selected such that:

$$\overbrace{[A^{l-1}B, \ldots, AB, B]}^{C_l} \overbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}}^{\vec{\omega}} = \overbrace{\vec{x}_{target} - A^l x[0]}^{\vec{z} \in \mathbb{R}^n}$$

By controllability, columnspace$(C_l) = \mathbb{R}^n$. $l \geq n$, so $\vec{\omega}$ exists. But if $l > n$, C_l is a wide matrix $\rightarrow \vec{\omega}$ has infinitely many solutions.

Minimum norm solution is a good choice b/c $\|\vec{\omega}\| = \sqrt{u[0]^2 + \cdots + u[l-1]^2}$ (interpret as "control energy") ex) longitudinal motion of car: $\frac{d}{dt}p(t) = v(t)$, $\frac{d}{dt}v(t) = \frac{1}{RM}u(t)$

$$\rightarrow \begin{bmatrix} p_d[i+1] \\ v_d[i+1] \end{bmatrix} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_d[i] \\ v_d[i] \end{bmatrix} + \begin{bmatrix} \frac{\Delta^2}{2RM} \\ \frac{\Delta}{RM} \end{bmatrix} u_d[i].$$

Controllable? i.e. are B & AB lin. indep.?

$B = \frac{\Delta}{RM}\begin{bmatrix} \frac{\Delta}{2} \\ 1 \end{bmatrix}$, $AB = \frac{\Delta}{RM}\begin{bmatrix} \frac{3}{2}\Delta \\ 1 \end{bmatrix}$ $\rightarrow$ Yes, linearly independent $\rightarrow$ Controllable

Suppose $\vec{x}[0] = \begin{bmatrix} p_d[0] \\ v_d[0] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\vec{x}_{target} = \begin{bmatrix} p_t \\ 0 \end{bmatrix} = [A^{l-1}B, \dots, AB, B]\begin{bmatrix} u[0] \\ \vdots \\ u[l-1] \end{bmatrix}$.

In theory, can find solution with $l=2 \rightarrow \begin{bmatrix} p_t \\ 0 \end{bmatrix} = [AB, B]\begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$.

$$\begin{bmatrix} u_d[0] \\ u_d[1] \end{bmatrix} = [AB, B]^{-1}\begin{bmatrix} p_t \\ 0 \end{bmatrix} \xrightarrow[\text{algebra}]{\text{skip}} \begin{bmatrix} u_d[0] \\ u_d[1] \end{bmatrix} = \frac{RM}{\Delta^2} p_{target}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Assume: $RM = 5000\,kg\cdot m$ ($R \simeq 0.3m$, $M \simeq 1600kg$), $\Delta = 0.1s$, $p_{target} = 1000m$.

$\rightarrow \begin{bmatrix} u_d[0] \\ u_d[1] \end{bmatrix} = 5 \times 10^8 \begin{bmatrix} 1 \\ -1 \end{bmatrix} [kg\frac{m^2}{s^2}], [N\cdot m]. \rightarrow$ impossible to implement!

More reasonable: $l = 1200$ ($l\Delta = 120s = 2min$)

$\begin{bmatrix} p_t \\ 0 \end{bmatrix} = \underbrace{[A^{1199}B, \dots, AB, B]}_{C_{1200}}\begin{bmatrix} u[0] \\ \vdots \\ u[1199] \end{bmatrix} \rightarrow$ minimum norm solution gives reasonable torque magnitudes

SVD: What is the rank of matrix $\vec{u}\vec{v}^T$, $\vec{u}\neq\vec{v}\neq 0$ (column x row)? $\rightarrow$ 1.

$\vec{u}\vec{v}^T = \vec{u}[v_1\ v_2 \cdots] = [v_1\vec{u}\ \ v_2\vec{u} \cdots] \rightarrow col(\vec{u}\vec{v}^T) = span(\vec{u})$

SVD seperates a rank r matrix $A \in \mathbb{R}^{m\times n}$ into a sum of rank 1 matrices, each written as outer products.

Specifically, we can find:

1) Orthonormal vectors $\vec{u}_1, \ldots, \vec{u}_r \in \mathbb{R}^m$, $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{R}^n$
2) Real, positive numbers $\sigma_1, \ldots, \sigma_r$ (singular values of A) s.t.

$\underline{A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T}$. By convention, singular values are put in decreasing order ($\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$).

This is the outer product form of SVD.

Compact form of SVD: $A = \underbrace{[\vec{u}_1, \ldots, \vec{u}_r]}_{U_r} \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_r^T \end{bmatrix} \rightarrow V_r^T$

$U_r \in \mathbb{R}^{m\times r}$, $V_r^T \in \mathbb{R}^{r\times n}$, and U_r & V_r have orthonormal columns.

$$A_{(j,k)} = \sum_{i=1}^{r} \overbrace{U_r(j,i)}^{(\vec{u}_i)_j} \cdot \sigma_i \cdot \overbrace{V_r^T(i,k)}^{\rightarrow V_r(k,i) = (\vec{v}_i)_k} = \sum_{i=1}^{r} \sigma_i (\vec{u}_i)_j (\vec{v}_i)_k$$

$\rightarrow$ This matches the (j,k)th entry in the outer product form.

ex1) $A=\begin{bmatrix}4&4\\3&3\end{bmatrix}$, $\operatorname{rank}(A)=\underline{r=1} \rightarrow A=\sigma_1\vec{u}_1\vec{v}_1^T=\begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}$

$=5\begin{bmatrix}4/5\\3/5\end{bmatrix}\cdot\sqrt{2}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=5\sqrt{2}\begin{bmatrix}4/5\\3/5\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\rightarrow \sigma_1=5\sqrt{2},\ \vec{u}_1=\begin{bmatrix}\frac{4}{5}\\\frac{3}{5}\end{bmatrix},\ \vec{v}_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

If we change signs of $\vec{u}_1$ and $\vec{v}_1$, length & outer product are same.

$\rightarrow$ another SVD

How to generalize to ≤ 2 ranks? Use evalues & evectors of

A^TA ($A\in\mathbb{R}^{m\times n}\rightarrow A^TA\in\mathbb{R}^{n\times n}$)

Some facts about A^TA:

① A^TA has real evalues/evectors $(\lambda_i,\vec{v}_i)$, $i=1,\cdots,n$

Proof (Spectral Theorem): A^TA is symmetric ($(A^TA)^T=A^T(A^T)^T=A^TA$)

② Evalues of A^TA are non-negative.

Proof: $A^TA\vec{v}_i=\lambda_i\vec{v}_i \xrightarrow[\vec{v}_i^T\cdot]{} \vec{v}_i^TA^TA\vec{v}_i=\lambda_i\vec{v}_i^T\vec{v}_i=\lambda_i\|\vec{v}_i\|^2$

$\rightarrow (A\vec{v}_i)^T(A\vec{v}_i)=\|A\vec{v}_i\|^2\rightarrow \lambda_i=\frac{\|A\vec{v}_i\|^2}{\|\vec{v}_i\|^2}\geq 0$

③ If $\operatorname{rank}(A)=r$, then r evalues of A^TA are strictly positive

Proof: first, note that $\operatorname{null}(A)=\operatorname{null}(A^TA)$.

i) $\operatorname{null}(A)\subseteq\operatorname{null}(A^TA)$: $A\vec{v}=\vec{0}\rightarrow A^TA\vec{v}=A^T0=0$ $\quad A\vec{v}=\vec{0}$

ii) $\operatorname{null}(A^TA)\subseteq\operatorname{null}(A)$: $A^TA\vec{v}=0 \xrightarrow[\vec{v}^T\cdot]{} \vec{v}^TA^TA\vec{v}=(A\vec{v})^T(A\vec{v})=\|A\vec{v}\|^2=0$

$\Rightarrow \operatorname{null}(A)=\operatorname{null}(A^TA)$.

$\mathrm{rank}(A)=r \rightarrow \dim(\mathrm{null}(A)) \overset{\text{rank-nullity theorem}}{=} n-r = \dim(\mathrm{null}(A^TA))$.

A^TA is $n\times n$, has $n-r$ dim. null space. Elements of $\mathrm{null}(A^TA)$ are e vectors of A^TA (for evalues of 0 repeated $(n-r)$ times)

② claims evalues are non-negative + $(n-r)$ evalues are zero

$\rightarrow$ remaining r evalues are positive. //

SVD Procedure for $A \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(A)=r$ (Using A^TA):

1) Find orthogonal matrix V diagonalizing A^TA

(V exists by Spectral Theorem)

$$\rightarrow V^T(A^TA)V = \begin{bmatrix} \lambda_1 & & & & 0 \\ \vdots & \ddots & & & \vdots \\ & & \lambda_r & & \\ & & & 0 & \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} \quad \begin{array}{l} \\ \\ \\ \\ ]\,n-r \end{array} \quad (\lambda_1,\dots,\lambda_r \text{ are evalues of } A^TA)$$

make sure that evalues are in decreasing order ($\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$)

2) For $i=1,\dots,r$, pick the ith column $\vec{v}_i$ of V (evector of A^TA for λ_i)

and let $\underline{\sigma_i = \sqrt{\lambda_i}}$*, $\underline{\vec{u}_i = \frac{1}{\sigma_i}A\vec{v}_i}$.*

ex2) $A=\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix}$, $\text{rank}(A)=\underline{r=2}$ $A^TA=\begin{bmatrix}4 & -3\\ 4 & 3\end{bmatrix}\begin{bmatrix}4 & 4\\ -3 & 3\end{bmatrix}=\begin{bmatrix}25 & 7\\ 7 & 25\end{bmatrix}$ 32, 18

$\lambda I-A=\begin{bmatrix}\lambda-25 & -7\\ -7 & \lambda-25\end{bmatrix}\rightarrow \det(\lambda I-A)=(\lambda-25)^2-7^2=0\rightarrow \lambda-25=\pm 7\rightarrow \lambda_{1,2}=25\pm 7$

$\underline{\lambda_1=32,\ \lambda_2=18}$ $\lambda_1 I-A^TA=\begin{bmatrix}7 & -7\\ -7 & 7\end{bmatrix}$, $\lambda_2 I-A^TA=\begin{bmatrix}-7 & -7\\ -7 & -7\end{bmatrix}$

$\underline{\vec{v}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix},\ \vec{v}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}}$

② $\underline{\sigma_1}=\sqrt{\lambda_1}=\sqrt{32}=\underline{4\sqrt{2}}$, $\underline{\vec{u}_1}=\frac{1}{4\sqrt{2}}A\vec{v}_1=\frac{1}{4\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}8\\ 0\end{bmatrix}=\underline{\begin{bmatrix}1\\ 0\end{bmatrix}}$

$\underline{\sigma_2}=\sqrt{\lambda_2}=\sqrt{18}=\underline{3\sqrt{2}}$, $\underline{\vec{u}_2}=\frac{1}{3\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ -6\end{bmatrix}=\underline{\begin{bmatrix}0\\ -1\end{bmatrix}}$

$\Rightarrow A=\underbrace{4\sqrt{2}}_{\sigma_1}\underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{u_1}\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}}_{v_1^T}+\underbrace{3\sqrt{2}}_{\sigma_2}\underbrace{\begin{bmatrix}0\\ -1\end{bmatrix}}_{u_2}\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}}_{v_2}$

Why does this procedure work? (Justification)

i.e. do $\vec{v}_i, \vec{u}_i, \sigma_i$ from procedure satisfy the following:

- $\sum_{i=1}^{r}\sigma_i \vec{u}_i \vec{v}_i = A$,
- $\vec{u}_1 \dots \vec{u}_r$ are orthonormal,
- $\vec{v}_1 \dots \vec{v}_r$ are orthonormal, → trivial by construction of V (step ①)
- $\sigma_1 \dots \sigma_r$ are real & positive → $\sigma_i=\sqrt{\lambda_i}$, λ_i are real and positive → $\sigma_i>0$

→ $\vec{u}_i^T\vec{u}_j=\begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases}$? $\vec{u}_i^T\vec{u}_j=\left(\frac{1}{\sigma_i}A\vec{v}_i\right)^T\left(\frac{1}{\sigma_j}A\vec{v}_j\right)=\frac{1}{\sigma_i\sigma_j}\vec{v}_i^T\underbrace{A^TA\vec{v}_j}_{\lambda_j\vec{v}_j}$,

$=\frac{\lambda_j}{\sigma_i\sigma_j}\vec{v}_i^T\vec{v}_j=\begin{cases}\frac{\lambda_j}{\sigma_i\sigma_j} & (i=j)\\ 0 & (i\neq j)\end{cases}$; if $i=j$, $\sigma_i\sigma_j=\sqrt{\lambda_j}^2=\lambda_j$

$\Rightarrow \vec{u}_i^T\vec{u}_j=\begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases}$ //

Proof of $\left(\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = A\right)$:

$\vec{u}_i \vec{v}_i^T = \left(\frac{1}{\sigma_i} A\vec{v}_i\right)\vec{v}_i^T = \frac{1}{\sigma_i} A\vec{v}_i\vec{v}_i^T \rightarrow \sum_{i=1}^{r} A\vec{v}_i\vec{v}_i^T = A$?

recall V orthogonal $\rightarrow V^TV = VV^T = I \rightarrow [\vec{v}_1 \cdots \vec{v}_n]\begin{bmatrix}\vec{v}_1^T \\ \vdots \\ \vec{v}_n^T\end{bmatrix} = \sum_{i=1}^{n} v_i v_i^T = I$

for $i > r$, $\vec{v}_i$ is evector of A^TA corresponding to a zero evalue

$\rightarrow A\vec{v}_i = 0 \rightarrow A\vec{v}_i\vec{v}_i^T = 0 \quad (r+1 < i \leq n)$

$\rightarrow \sum_{i=r+1}^{n} A\vec{v}_i\vec{v}_i^T = 0, \quad A\sum_{i=1}^{n} \vec{v}_i\vec{v}_i^T = A$

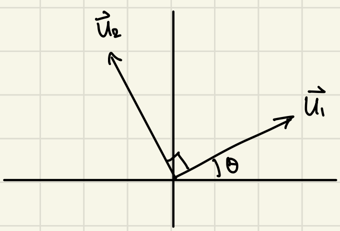
$\Rightarrow A\sum_{i=1}^{r} \vec{v}_i\vec{v}_i^T = A$ //

ex3) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \lambda_{1,2} = 1, -1$, $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \sigma_1 = \sigma_2 = 1$ for A

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $U = I$ works for step 1 $\rightarrow \lambda_1 = \lambda_2 = 1$

$\vec{v}_1 = A^T\vec{u}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Since $AA^T = I$, any orthonormal $(\vec{u}_1, \vec{u}_2)$

will work. $\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. $\rightarrow$ This is more general.



$\vec{v}_1 = A^T\vec{u}_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}, \vec{v}_2 = A^T\vec{u}_2 = \begin{bmatrix} -\sin\theta \\ -\cos\theta \end{bmatrix}$

$\rightarrow$ repeated evalues are another source of nonuniqueness.

$AA^T \in \mathbb{R}^{m\times m}$ has same claims for A^TA; real positive r # of evalues,

remaining (m-r) are zero.

① Find orthogonal matrix U s.t. $U \in \mathbb{R}^{m\times m}$ diagonalizes A^TA

$$U^TAA^TU = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} \Big] m-r$$

② For each $i = 1, \dots, r$, take i-th column $\vec{u_i}$ of U ($A\vec{u_i} = \lambda_i \vec{u_i}$)

Let $\sigma[i] = \sqrt{\lambda_i}$, $\vec{v_i} = \frac{1}{\sigma_i} A^T \vec{u_i}$

ex) $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \rightarrow AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

Choose $\vec{u_i} = I$, $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \lambda_1 = 32$, $\lambda_2 = 18$

$\rightarrow \sigma_1 = 4\sqrt{2}$, $\sigma_2 = 3\sqrt{2} \rightarrow \vec{v_1} = \frac{1}{\sigma_1} A^T \vec{u} + \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v_2} = \frac{1}{\sigma_2} A^T \vec{u_2} = \frac{1}{3\sqrt{2}} \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \vec{u_2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{v_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Full SVD: $A = \underbrace{[\vec{u}_1 \cdots \vec{u}_r]}_{\substack{U_r \\ m\times r}} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{\substack{\Sigma_r \\ r\times r}} \underbrace{\begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_r^T \end{bmatrix}}_{\substack{V_r^T \\ r\times n}}$ Σ_r → make complete $\vec{u}_1 \cdots \vec{u}_r$ in an orthonormal basis $\in \mathbb{R}^m$ with $\vec{u}_{r+1} \cdots \vec{u}_m$, same for $\vec{v}$

$$\rightarrow \underbrace{[\overbrace{\vec{u}_1 \cdots \vec{u}_r}^{U_r} \,\vdots\, \overbrace{\vec{u}_{r+1} \cdots u_m}^{U_{m-r}}]}_{m\times m} \underbrace{\left[\begin{array}{c:c} \begin{matrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{matrix} & \vec{0}_{r\times(n-r)} \\ \hdashline \vec{0}_{(m-r)\times r} & \vec{0}_{(m-r)\times(n-r)} \end{array}\right]}_{\Sigma \quad m\times n} \underbrace{\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \\ \hdashline \vec{v}_{r+1} \\ \vdots \\ \vec{v}_n \end{bmatrix}}_{n\times n} \begin{matrix} \rightarrow V_r^T \\ V^T \rightarrow \text{orthogonal} \\ V_{n-r}^T \end{matrix}$$

$U \rightarrow$ orthogonal

$\rightarrow A = U \Sigma V^T$

Notes:

① If A is wide and full row rank $(n > m = r)$, $\Sigma = [\Sigma_r \quad 0_{r\times(n-r)}]$

If A is tall and full column rank, $\Sigma = \begin{bmatrix} \Sigma_r \\ 0_{(m-r)\times n} \end{bmatrix}$

If A is square and full rank, $\Sigma = \Sigma_n$

② $\vec{v}_{r+1} \cdots \vec{v}_n$ are evectors of A^TA corresponding to 0 evalues (orthonormal basis for $\text{Null}(A^TA) = \text{null}(A)$)

$\vec{u}_{r+1} \cdots \vec{u}_m$ are evectors for 0 evalues of AA^T. Therefore, an orthonormal basis for $\text{Null}(AA^T) = \text{null}(A^T)$

$\rightarrow \text{Col}(V_{n-r}) = \text{Null}(A)$, $\text{Col}(U_{m-r}) = \text{Null}(A^T)$.

Similarly, $A\vec{x} = \sum_{i=1}^{r} \sigma_i \vec{u}_i \underline{\vec{v}_i^T \vec{x}} = \sum_{i=1}^{r} \underbrace{\sigma_i (\vec{v}_i^T \vec{x})}_{\text{scalar}} \vec{u}_i$

$\rightarrow$ In the span of $\vec{u}_1 \cdots \vec{u}_r$, $\text{Col}(A) = \text{Col}(U_r)$.

Similary, $\text{Col}(A^T) = \text{Col}(V_r)$.

ex1) $A=\begin{bmatrix}4&4\\3&3\end{bmatrix}=5\sqrt{2}\left(\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\end{bmatrix}\right)\rightarrow \sigma_1=5\sqrt{2},\ \vec{u}_1=\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix},\ \vec{v}_1^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\end{bmatrix}$

$\rightarrow$ Put it in full SVD form: $\vec{u}_2=\frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix},\ \vec{v}_2^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\end{bmatrix}$

$\rightarrow U=[\vec{u}_1\ \vec{u}_2]=\frac{1}{5}\begin{bmatrix}4&-3\\3&4\end{bmatrix},\ V=[\vec{v}_1\ \vec{v}_2]=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

$\rightarrow A=U\begin{bmatrix}5\sqrt{2}&0\\0&0\end{bmatrix}V^T\rightarrow$ $\underline{\vec{u}_1 \text{ spans } Col(A),\ \vec{v}_1 \text{ spans } Col(A^T).}$

$\underline{\vec{u}_2 \text{ spans } Null(A^T),\ \vec{v}_2 \text{ spans } Null(A).}$

Geometric Interpretation of SVD: $A=U\Sigma V^T$

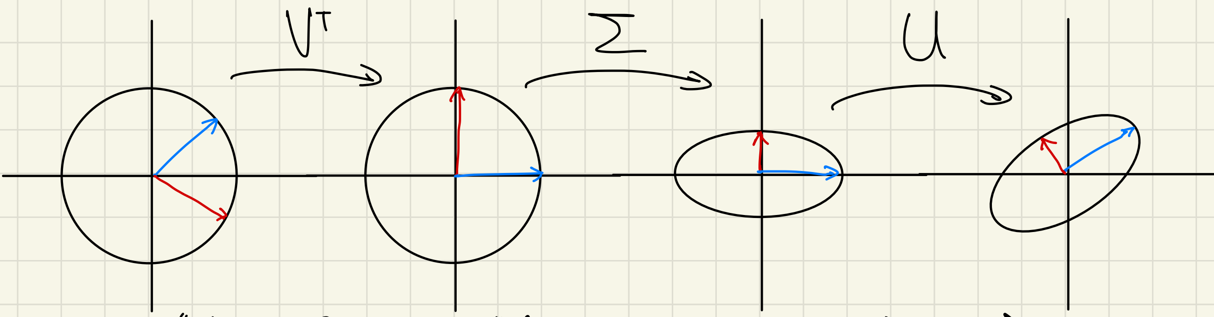
U, V are orthogonal $\rightarrow \|U\vec{x}\|=\|\vec{x}\|$ $\left(\|U\vec{x}\|^2=(U\vec{x})^T(U\vec{x})=\vec{x}^T\cancel{U^TU}\vec{x}=\|\vec{x}\|^2\right)$

i.e. multiplication by orthogonal matrices do not change length!

Also, multiplying a vector by $\Sigma_r=\begin{bmatrix}\sigma_1&&\\&\ddots&\\&&\sigma_r\end{bmatrix}$ stretches first entry by σ_1, second by σ_2, and so on.

$A\vec{x}=U\Sigma V^T\vec{x}.$

① $V^T\vec{x}$ ② $\Sigma(V^T\vec{x})$ ③ $U(\Sigma V^T\vec{x})$



$\|A\vec{x}\|\leq\sigma_1\|\vec{x}\|,\ \|A\vec{x}\|=\sigma_1\|\vec{x}\|$ when $\vec{x}=\alpha\vec{v}_1$

Applications of SVD

Suppose $m=n=r$ for $A \in \mathbb{R}^{m\times n} \to A$ is invertible.

$$A = U\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{bmatrix}V^T. \quad A^{-1} = V\begin{bmatrix}1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n\end{bmatrix}U^T. \quad (AA^{-1} = U\Sigma V^T V \Sigma^{-1} U^T = I)$$

Thus, SVD makes inversion easy. Also, a "pseudo inverse" can be derived from SVD when inverse does not exist.

Define: Given $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A) = r$, $A = U\underbrace{\begin{bmatrix}\Sigma_r & \vec{0} \\ \vec{0}^T & [0]\end{bmatrix}}_{m\times n}V^T$,

the <u>Moore-Penrose Pseudoinverse</u> is given by:

$$A^+ = V\underbrace{\begin{bmatrix}\Sigma_r^{-1} & 0 \\ 0 & [0]\end{bmatrix}}_{n\times m}U^T. \ * \quad \text{Or, equivalently, } A^+ = V_r\Sigma_r^{-1}U_r^T = \sum_{i=1}^{r}\frac{1}{\sigma_i}\vec{v}_i\vec{u}_i^T.$$

ex) $A = \begin{bmatrix}1 \\ 2\end{bmatrix}$. $A = \overset{\sigma_1}{\sqrt{5}}\overset{\vec{u}_1}{\begin{bmatrix}1/\sqrt{5} \\ 2/\sqrt{5}\end{bmatrix}}\overset{\vec{v}_1^T}{[1]} \to A^+ = \frac{1}{\sqrt{5}}[1]\begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix} = \underline{\begin{bmatrix}1/5 & 2/5\end{bmatrix}}$.

Note: ① $A^+ \in \mathbb{R}^{n\times m}$ when $A \in \mathbb{R}^{m\times n}$. ② Applies to any $A \neq 0$.

③ If $m=n=r$, $A^+ = V\Sigma^{-1}U^T = A^{-1}$ (as shown above).

④ $AA^+ = (U_r\Sigma_rV_r^T)(V_r\Sigma_r^{-1}U_r^T) = U_r\Sigma_r\Sigma_r^{-1}U_r^T = U_rU_r^T = \operatorname{proj}_{(\operatorname{Col}(A))}(\cdot)$

⑤ $A^+A = (V_r\Sigma_r^{-1}U_r^T)(U_r\Sigma_rV_r^T) = V_r\Sigma_r^{-1}\Sigma_rV_r^T = V_rV_r^T = \operatorname{proj}_{(\operatorname{Col}(A^T))}(\cdot)$

$QQ^T\vec{x} = [\vec{q}_1 \cdots \vec{q}_k]\begin{bmatrix}\vec{q}_1^T\vec{x} \\ \vdots \\ \vec{q}_k^T\vec{x}\end{bmatrix} = (\vec{q}_1^T\vec{x})\vec{q}_1 + \cdots + (\vec{q}_k^T\vec{x})\vec{q}_k \to$ projects $\vec{x}$ onto $\operatorname{Col}(Q)$

Least Squares with SVD: Want to minimize $\|A\vec{x}-\vec{y}\|$ when $m>n$ (tall).

Recall the minimizer $\vec{x}_{LS}$ is s.t. $A\vec{x}_{LS} = \text{proj}_{(\text{Col}(A))}\vec{y} = U_r U_r^T \vec{y} = AA^\dagger \vec{y}$.

$\rightarrow A\vec{x}_{LS} = AA^\dagger\vec{y}. \Rightarrow \underline{\vec{x}_{LS} = A^\dagger \vec{y}}$. Difference to $\vec{x}_{LS} = (A^TA)^{-1}A^T\vec{y}$?

It's the same b/c $\underline{A^\dagger = (A^TA)^{-1}A^T}$ when $\underline{A \text{ has full column rank}}$ $(r=n)$

ex) $A = \begin{bmatrix}1\\2\end{bmatrix}$. $A^TA = 5 \rightarrow A^\dagger = \frac{1}{5}\begin{bmatrix}1 & 2\end{bmatrix}$ (same result!) Why?

Proof: substitute SVD into $(A^TA)^{-1}A^T = A^\dagger$. $A = U_r\Sigma_r V_{(r)}^T$. ($V_r = V$ b/c $r=n$)

$\rightarrow (A^TA)^{-1}A^T = (V\Sigma_r^{(T)}\cancel{U_r^T U_r}^{I}\Sigma_r V^T)^{-1}(V\Sigma_r^{(T)}U_r^T)$

(compact form)

$= (V\Sigma_r^2 V^T)^{-1} V\Sigma_r U_r^T = V\Sigma_r^{(-2)}\cancel{V^T V}^{I}\Sigma_r U_r^T = V\Sigma_r^{-1}U_r^T = A^\dagger$. //

Similarly, when A is wide and has $\underline{\text{full row rank}}$ $(r=m)$,

$\underline{A^\dagger = A^T(AA^T)^{-1}}$ can be shown by derivation like above.

Minimum Norm Solution: $m<n$ (wide matrix), so $A\vec{x}=\vec{y}$ has infinite sol'n.

want $\vec{x}$ with least $\|\vec{x}\|$. Substitute compact SVD into $A\vec{x}=\vec{y}$:

$U_r\Sigma_r V_r^T\vec{x} = \vec{y}. \rightarrow \cancel{U_r^T U_r}^{I}\Sigma_r V_r^T\vec{x} = U_r^T\vec{y}. \rightarrow V_r^T\vec{x} = \Sigma_r^{-1}U_r^T\vec{y}$.

Any $\vec{x}$ satisfying $\underline{V_r^T\vec{x} = \Sigma_r^{-1}U_r^T\vec{y}}$ is a solution to $A\vec{x}=\vec{y}$.

Use $\|\vec{x}\| = \|V^T\vec{x}\| = \left\|\begin{bmatrix}V_r^T\\V_{n-r}^T\end{bmatrix}\vec{x}\right\| = \left\|\begin{bmatrix}V_r^T\vec{x}\\V_{n-r}^T\vec{x}\end{bmatrix}\right\|$, and the top part is fixed.

Set $V_{n-r}^T\vec{x} = 0$ to minimize $\|\vec{x}\|$. (makes $\vec{x}$ orth. to Null(A)!) $\rightarrow \begin{bmatrix}V_r^T\vec{x}\\V_{n-r}^T\vec{x}\end{bmatrix} = \begin{bmatrix}\Sigma_r^{-1}U_r^T\vec{y}\\0\end{bmatrix} = V^T\vec{x}$.

$\rightarrow VV^T\vec{x} = V\begin{bmatrix}\Sigma_r^{-1}U_r^T\vec{y}\\0\end{bmatrix} \rightarrow \vec{x}_{mn} = \begin{bmatrix}V_r & V_{n-r}\end{bmatrix}\begin{bmatrix}\Sigma_r^{-1}U_r^T\vec{y}\\0\end{bmatrix} = V_r\Sigma_r^{-1}U_r^T\vec{y} = \underline{A^\dagger\vec{y}}$.

ex) controllability. $\underbrace{\vec{X}_{target} - A^{\ell}\vec{X}[0]}_{\vec{y}} = \underbrace{[A^{\ell-1}B \cdots AB \;\; B]}_{C_\ell} \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$

Solution exists by controllability. Use $\vec{x}_{MN} = A^T(AA^T)^{-1}\vec{y}$.

$$\begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}_{MN} = C_L^T(C_LC_L^T)^{-1}(\vec{X}_{target} - A^{\ell}\vec{X}[0])$$

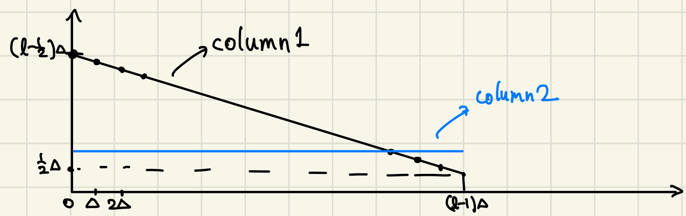
For the vehicle control example: $A = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}$, $B = \frac{\Delta}{RM}\begin{bmatrix} \frac{\Delta}{2} \\ 1 \end{bmatrix}$.

$AB = \frac{\Delta}{RM}\begin{bmatrix} \frac{3}{2}\Delta \\ 1 \end{bmatrix}$, $A^2B = \frac{\Delta}{RM}\begin{bmatrix} \frac{5}{2}\Delta \\ 1 \end{bmatrix} \cdots A^{\ell-1}B = \begin{bmatrix} (\ell-\frac{1}{2})\Delta \\ 1 \end{bmatrix}$.

$C_\ell = \frac{\Delta}{RM}\begin{bmatrix} (\ell-\frac{1}{2})\Delta & \cdots & \frac{3}{2}\Delta & \frac{1}{2}\Delta \\ 1 & \cdots & 1 & 1 \end{bmatrix}$. $\vec{X}[0] = \vec{0}$, $\vec{X}_{target} = \begin{bmatrix} 1000 \\ 1 \end{bmatrix}$, $\Delta = 0.1s$, $RM = 5000$.

$\vec{U}_{MN} = \frac{\Delta}{RM}\begin{bmatrix} (\ell-\frac{1}{2})\Delta & 1 \\ \vdots & \vdots \\ \frac{1}{2}\Delta & 1 \end{bmatrix} \underbrace{(C_\ell C_\ell^T)^{-1}\begin{bmatrix} 1000 \\ 0 \end{bmatrix}}_{(2\times 1)}$ → weighted sum of colums of C_ℓ^T (rows of C_ℓ)

⇒ Min. norm control sequence is a linear combination of two sequences.



Low Rank Approximation: Given a high-rank matrix $A \in \mathbb{R}^{m \times n}$, $r \simeq \min(m,n)$, can we find an approximation for A with rank $\ell \ll \min(m,n)$?

SVD (in outer product form) $\rightarrow$ $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = \underbrace{\sum_{i=1}^{\ell} \sigma_i \vec{u}_i \vec{v}_i^T}_{=: A_\ell} + \overbrace{\cancel{\sum_{i=(\ell+1)}^{r} \sigma_i \vec{u}_i \vec{v}_i^T}}^{\simeq 0}$

Note: A_ℓ in outer product form has far fewer entries to store than A.

ex) $A \in \mathbb{R}^{1000 \times 1000}$, A has 10^6 entries. A_ℓ has $(1000+1000)\ell$ entries.

Also, in many datasets, a few singular values are dominant & rest are small.

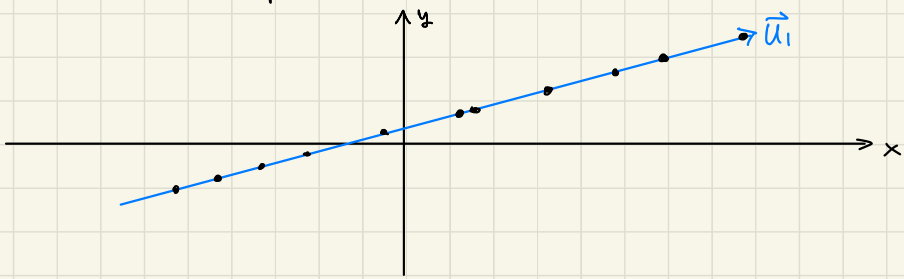
Therefore, there is not much loss when truncated.

Eckart-Young Theorem states that SVD truncation produces the rank ℓ matrix with the least deviation from original A, as measured by Frobenius Norm: $\min_{B \in \mathbb{R}^{m \times n}} \|A - B\|_F$ s.t. $\mathrm{rank}(B) = \ell$,

A_ℓ above solves the optimization problem.

Principal Component Analysis (PCA):

Suppose A has $m=2$ rows and many more columns, $n \gg 2$. If $\underline{r=1}$, what does the scatterplot of columns of A look like?

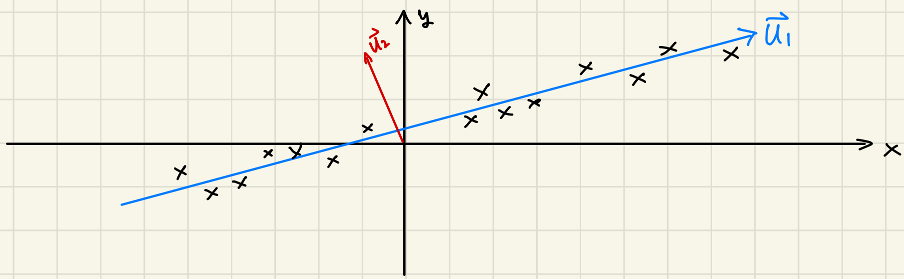


$A=\sigma_1 \vec{u}_1 \vec{v}_1^T \to \vec{a}_1=(\sigma_1 \vec{v}_1^T(1))\vec{u}_1,\ \vec{a}_i=(\sigma_1 \vec{v}_1^T(i))\vec{u}_1.$

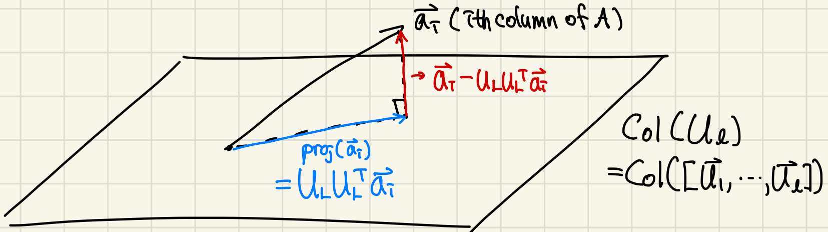
What if $\underline{r=2}$, but $\sigma_1 \gg \sigma_2$? $A=\sigma_1 \vec{u}_1 \vec{v}_1^T+\sigma_2 \vec{u}_2 \vec{v}_2^T$.

$\vec{a}_i=\sigma_1 \underline{\vec{v}_1^T(i)}\,\vec{u}_1+\sigma_2 \underline{\vec{v}_2^T(i)}\,\vec{u}_2$. $\vec{v}_1(i)$ and $\vec{v}_2(i)$ are generally in the same range.

However, $\sigma_1 \vec{v}_1^T(i) \gg \sigma_2 \vec{v}_2^T(i)$ since $\sigma_1 \gg \sigma_2$.



For a general matrix with ℓ dominant singular values, we expect the scatterplot of columns to cluster around the subspace spanned by $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_\ell\}$.

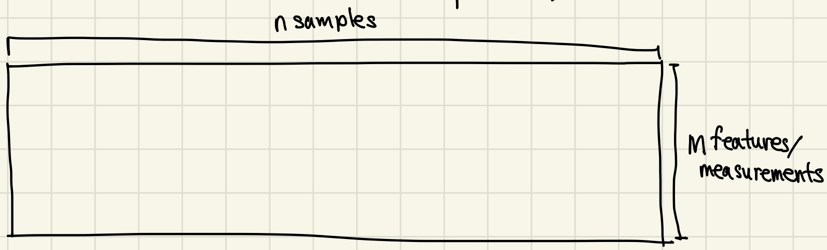


Distance to subspace(U_L) is $\underline{\|\vec{a}_i - U_L U_L^T \vec{a}_i\|}$.

Use sum of squared distance as a measure of closeness to sub(U_i):

$\sum_{i=1}^{n} \|\vec{a}_i - U_L U_L^T \vec{a}_i\|^2$. By Theorem 4 in Note 17, if we pick another matrix W with ℓ orthonormal columns, $\sum_{i=1}^{n} \|\vec{a}_i - WW^T \vec{a}_i\|^2 \geq \sum_{i=1}^{n} \|\vec{a}_i - U_L U_L^T \vec{a}_i\|^2$.

PCA - when A is a collection of data points,

$A =$ 

(e.g. $n = 1000$ 16B students, $m =$ HW scores)

the vectors $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r$ in SVD are called "principal components of A".

Thus, PCA shows directions of dominant values in datasets. The first few give a lower dimension representation of data. Evectors of $m \times m$ (AA^T) are the principal components, ordered by $\lambda_1 = \sigma_1^2, \ldots$.

Subtract from each row the average of it. Then,

$\left(\frac{1}{n-1} AA^T\right)$ is the covariance matrix. $\frac{1}{n-1} AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a, d are $(\text{stdev})^2$.

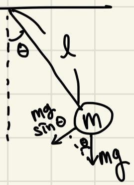
Linearization

Nonlinear Systems: $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, $\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i])$

where $\vec{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector-valued functions of $\vec{x} \in \mathbb{R}^n$, $\vec{u} \in \mathbb{R}^m$.

Linear systems are a special case s.t. $\vec{f}(\vec{x}, \vec{u}) = A\vec{x} + B\vec{u}$.

ex1) Pendulum model: $ml\frac{d^2\theta}{dt} = -mg\sin\theta - kl\frac{d\theta}{dt}$. $x_1 = \theta(t)$, $x_2 = \omega(t)$.



Let $\vec{x}(t) = \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$ where $\omega(t) = \frac{d\theta}{dt}$. $\to \frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} \omega(t) \\ \frac{d^2\theta}{dt}(t) \end{bmatrix}$

$$= \begin{bmatrix} \omega(t) \\ -\frac{k}{l}\omega - \frac{g}{l}\sin\theta \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{l}x_2(t) - \frac{g}{l}\sin(x_1(t)) \end{bmatrix} = \vec{f}(\vec{x}(t))$$

Equilibrium (Operating) Points: For a continuous system with no input, $\frac{d}{dt}\vec{x}(t) = \vec{f}_c(\vec{x}(t))$, the solutions of $\vec{f}_c(\vec{x}) = 0$ are eq. points.

If $\vec{x}_*$ is an equilibrium, then $\vec{x}(t) = \vec{x}_*$ is a solution of diff. eq. with condition $\vec{x}(0) = \vec{x}_*$ ($\frac{d}{dt}\vec{x}(t) = 0 \to \vec{x}(t) = C \to \vec{x}(t) = \vec{x}_*$).

Pendulum: $\vec{f}(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_2 - \frac{g}{l}\sin(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \underline{x_2 = 0,\ x_1 = 0, \pi} \to$ two eq. points

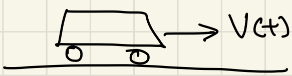
$(x_1, x_2) = (0, 0)$ (downward pointing), $(x_1, x_2) = (\pi, 0)$ (upward pointing).

Discrete Time Equilibria: $\vec{x}[i+1] = \vec{f}_d(\vec{x}[i])$. $\vec{x}_*$ is an equilibrium if $\vec{f}_d(\vec{x}_*[i]) = \vec{x}_*$. $\to \vec{x}[i] = \vec{x}_*$ is a solution of diff. eq. $\to \vec{x}_* = \vec{f}_d(\vec{x}_*[i])$

Systems with inputs: $\frac{d}{dt}\vec{x}(t) = f_c(\vec{x}(t), \vec{u}(t))$. $(\vec{x}^*, \vec{u}^*)$ is an "operating point" of system if $f_c(\vec{x}_*, \vec{u}_*) = 0$.

Discrete time: $(\vec{x}_*[i], \vec{u}_*[i])$ is an "operating point" for $\vec{x}[i+1] = \vec{f}_d(\vec{x}[i], \vec{u}[i])$ if $f_d(\vec{x}_*, \vec{u}_*) = \vec{x}_*$.

If we apply constant input $\vec{u}(t) = \vec{u}_*$ then $\vec{x}(t) = \vec{x}_*$ is a solution with $\vec{x}(0) = \vec{x}_*$

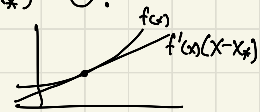
ex2) Vehicle: 

$M\frac{dV(t)}{dt} = -\beta V^2(t) + \frac{1}{R}u(t)$.

$x = V$ is state, $f(x,u) = -\frac{\beta}{M}x^2 + \frac{1}{RM}u$. (x_*, u_*) is an operating point if $-\frac{\beta}{M}x_*^2 + \frac{1}{RM}u_* = 0 \to u_* = \beta R x_*^2$ (want speed x_*, must apply torque u_*.)

Linearization: linear approximation of nonlinear model around operating point.

Easy when $x \in \mathbb{R}$: 1) No input $\to \frac{d}{dt}x(t) = f(x(t))$, $f(x_*) = 0$.

Taylor approximation: $f(x) \simeq f(x_*) + f'(x_*)(x - x_*)$ 

Define $\delta x(t) = x(t) - x^*$. $\frac{d}{dt}\delta x(t) = \frac{d}{dt}(x(t) - x_*) = \frac{d}{dt}x(t) \simeq f'(x_*)\delta x(t)$

Linearized model: $\frac{d}{dt}\delta x(t) = \overbrace{f'(x^*)}^{\lambda}\delta x(t)$

2) with input $u \in \mathbb{R}$: $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, $f(\vec{x}_*(t), \vec{u}_*(t)) = 0$.

$$f(x,y) \simeq f(x_*, y_*) + \underbrace{\frac{\partial f}{\partial x}(x_*, u_*)}_{\lambda}\underbrace{(x - x_*)}_{\delta x} + \underbrace{\frac{\partial f}{\partial y}(x_*, y_*)}_{b}\underbrace{(y - y_*)}_{\delta u}$$

$$\frac{d}{dt}\delta x(t) = \frac{d}{dt}x_*(t) = \lambda \delta x(t) + b\,\delta u(t).$$

ex2) $f(x,u) = -\frac{\beta}{M}x^2 - \frac{1}{RM}u$. $\frac{\partial f}{\partial x} = -2\frac{\beta x}{M}$. $\frac{\partial f}{\partial u} = -\frac{1}{RM}$. $\underline{\lambda = \frac{-2\beta x^*}{m},\ b = -\frac{1}{RM}}$.

$u_* = \beta R x_*^2$. Assume $\delta u = 0$ ($u(t) = u^*$). $\frac{d}{dt}\delta x(t) = \lambda \delta x(t)$.

$\to \delta x(t) = e^{\lambda t}\delta x(0)$. $\lambda < 0$ above, so $\delta x(t) \to 0$, i.e. $x(t) \to x_*$.

If $\lambda < 0$ is not negative enough (slow convergence to x_*), apply feedback — $\delta u(t) = k\,\delta x(t)$.

Closed-loop system: $\frac{d}{dt}\delta x(t) = (\lambda + bk)\delta x(t)$. Chooses k value to make $(\lambda + bk)$ as negative as we want $\to \delta x(t) = e^{(\lambda + bk)t}\delta x(0)$.

$$u(t) = u^* + \delta u(t) = \beta R x_*^2 + k\delta x(t) = \beta R x_*^2 + k(x(t) - x_*)$$

Next, assume $\vec{x} \in \mathbb{R}^2$, $u \in \mathbb{R}$, $\vec{f}(\vec{x}, u) \in \mathbb{R}^2$. $\vec{f}(\vec{x}, u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}$.

$$f_i(x_1, x_2, u) = \cancelto{0}{f_i(x_{1*}, x_{2*}, u_*)} + \frac{\partial f_i}{\partial x_1}\Big|_{(\vec{x}_*, u_*)} \cdot (x_1 - x_{1*}) + \frac{\partial f_i}{\partial x_2}\Big|_{(\vec{x}_*, u_*)}(x_2 - x_{2*}) + \frac{\partial f_i}{\partial u}\Big|_{(\vec{x}_*, u_*)}(u - u_*)$$

$$\to \frac{d}{dt}\delta\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\vec{x}_*, u_*)} \begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_1}{\partial u} \end{bmatrix}_{(\vec{x}_*, u_*)} \delta u(t) = J_{\vec{x}}\vec{f}(\vec{x}_*, u_*)\delta\vec{x}(t) + J_u\,\delta u(t)$$

Given nonlinear system $\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t))$, $\vec{f}(\vec{x}^*, \vec{u}^*) = \vec{0}$,

linearized model around operating point is:

$$\frac{d}{dt}\delta\vec{x}(t) = \overbrace{J_{\vec{x}}\vec{f}(\vec{x}^*, \vec{u}^*)}^{A} \cdot \delta\vec{x}(t) + \overbrace{J_{\vec{u}}\vec{f}(\vec{x}^*, \vec{u}^*)}^{B} \cdot \delta\vec{u}(t). \quad * \left(A(i,j) = \frac{\partial f_i}{\partial x_j}\Big|_{\vec{x}^*, \vec{u}^*}\right)$$

ex) Pendulum model: $x_1(t) = \theta(t)$, $x_2(t) = \frac{d\theta}{dt}(t) \rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin(x_1(t)) \end{bmatrix}$

denote: $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin(x_1)$. (eliminated time)

downward pointing equilibrium: $(x_1, x_2) = (0, 0)$.

upward 〃 : $(x_1, x_2) = (\pi, 0)$.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \left(-\frac{g}{\ell}\cos(x_1)\right) & \left(-\frac{k}{m}\right) \end{bmatrix} = J_{\vec{x}}\vec{f}(\vec{x}). \quad A = J_{\vec{x}}\vec{f}(\vec{x}^*)$$

$$A_{down} = J_{\vec{x}}\vec{f}(0,0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}, \quad A_{up} = J_{\vec{x}}\vec{f}(\pi,0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}$$

Linear model: $\frac{d}{dt}\delta\vec{x}(t) = A_{down}\delta\vec{x}(t)$, $\delta\vec{x} = \vec{x} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Stability criteria for (2x2) A matrices (continuous): $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det(\lambda I - A) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = \lambda^2 + tr(A)\lambda + \det(A).$$

$$\lambda_{1,2} = \frac{tr(A) \mp \sqrt{tr(A)^2 - 4\det(A)}}{2}.$$

(i) $\det(A) < 0 \rightarrow$ unstable, done.

(ii) $tr(A) > 0 \rightarrow$ unstable, done. (iii) $\det(A) > 0$ && $tr(A) < 0 \rightarrow$ stable!

$A_{down} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix}$. $tr(A_{down}) = -k/m$, $\det(A_{down}) = g/\ell$. $\rightarrow$ stable!

$A_{up} = \begin{bmatrix} 0 & 1 \\ g/\ell & -k/m \end{bmatrix}$. $tr(A_{up}) = -k/m$, $def(A_{up}) = -g/\ell$ $\rightarrow$ unstable!

Looking at evalues for A_{up} for further insights: $A_{up} = \begin{bmatrix} 0 & 1 \\ g/l & -k/m \end{bmatrix}$

$\det(\lambda I - A_{up}) = \lambda^2 + \frac{k}{m}\lambda - g/l \rightarrow \lambda_{1,2} = \dfrac{-k/m \mp \sqrt{\frac{k^2}{m^2} + 4\,g/l}}{2}$.

Instability is more severe when $g\uparrow$, $l\downarrow$ (λ_2 is more positive).

Linearization in Discrete time: $\vec{x}[i+1] = \vec{f}(\vec{x}[i], \vec{u}[i])$, $\vec{x}^* = \vec{f}(\vec{x}^*, \vec{u}^*)$.

$\vec{f}(\vec{x},\vec{u}) \simeq \overbrace{\vec{f}(\vec{x}^*, \vec{u}^*)}^{\vec{x}^*} + A\overbrace{(\vec{x} - \vec{x}^*)}^{\delta\vec{x}} + B\overbrace{(\vec{u} - \vec{u}^*)}^{\delta\vec{u}}$. plug into $\vec{f}(\vec{x}[i], \vec{u}[i])$.

$\rightarrow \vec{x}[i+1] = \vec{x}^* + A\delta\vec{x}[i] + B\delta\vec{u}[i]$.

$\rightarrow \underline{\delta\vec{x}[i+1] = A\delta\vec{x}[i] + B\delta\vec{u}[i]}$. //

ex) Population Growth model: $x[i+1] = r x[i]$. $\vec{x}[i]$:= population in generation i

if $r > 1 \rightarrow$ exponential growth...? limited resources, not realistic.

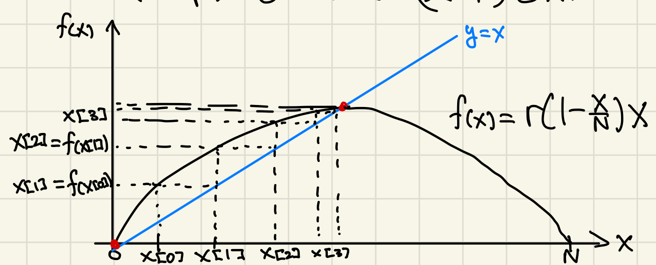
More realistic model: $x[i+1] = \underbrace{r\left(1 - \frac{x[i]}{N}\right)\vec{x}[i]}_{f(\vec{x}[i])}$. $\rightarrow$ nonlinear in $\vec{x}[i]$.

$f(x) = r\left(1 - \frac{x}{N}\right)\cdot x$. equilibrium point: $f(x) = x \rightarrow r\left(1 - \frac{x}{N}\right)x = x \rightarrow \underline{x = \left(N - \frac{N}{r}\right), 0}$

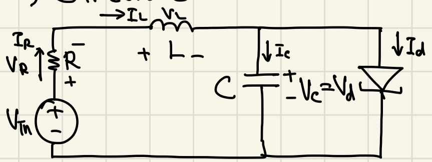
$f'(x) = r - 2\frac{r}{N}x \rightarrow f'(0) = r$, $f'\left(N - \frac{N}{r}\right) = r - 2r\left(1 - \frac{1}{r}\right) = \underline{2 - r}$.

Linearized model: around $x^* = 0$: $\delta x[i+1] = r\,\delta x[i]$ ($r > 1 \rightarrow$ unstable)

around $x^* = \left(N - \frac{N}{r}\right)$: $\delta x[i+1] = (2 - r)\,\delta x[i+1]$ (stable if $1 < r < 3$)



ex) Circuits



KCL: $I_R = I_L = I_c + I_d$. KVL: $V_{in} = V_R + V_L + V_c$, $V_c = V_d$.

$\vec{x} := \begin{bmatrix} V_c \\ I_L \end{bmatrix}$. $C\frac{dV_c}{dt} = I_c \rightarrow C\frac{dV_c}{dt} = I_L - I_d = I_L - g(V_d) = I_L - g(V_c)$.

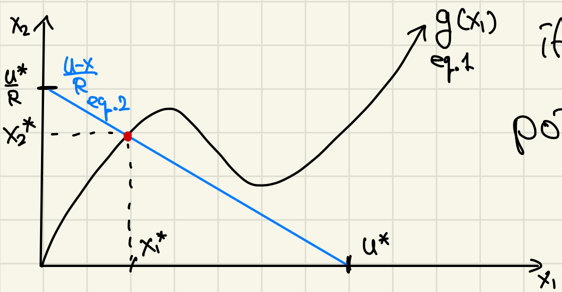
$L\frac{dI_L}{dt} = V_L \rightarrow L\frac{dI_L}{dt} = V_{in} - V_R - V_c = V_{in} - I_R R - V_c = V_{in} - I_L R - V_c$.

$\rightarrow \frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} V_c(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{C}(I_L - g(V_c)) \\ \frac{1}{L}(V_{in} - I_L R - V_c) \end{bmatrix} \rightarrow \frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t)) = \begin{bmatrix} \frac{1}{C}(x_2 - g(x_1)) \\ \frac{1}{L}(u - Rx_2 - x_1) \end{bmatrix}$

Operating points: $\vec{f}(\vec{x}, t) = \vec{0}$. $\rightarrow x_2 = g(x_1)$, $x_2 = \frac{u - x}{R} \rightarrow g(x_1) = \frac{u - x}{R}$

$\rightarrow$ Find (x_1^*, u^*) satisfying $g(x) = \frac{u - x}{R}$. $\rightarrow x_2^* = g(x_1^*) \rightarrow \underline{(x_1^*, x_2^*, u^*)}$

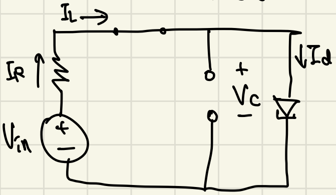
Graphical interpretation: superimpose two equations for x_2.



if we increase $u^*(V_{in})$, equilibrium points increase to 3 then back to 1.

Circuit interpretation: if $\vec{f}(\vec{x}^*, u^*) = \vec{0}$, $\frac{d}{dx}\vec{x}(t) = 0$ when $\begin{cases} \vec{x}(0) = \vec{x}^* \\ u(t) = u^* \end{cases}$.

in this circuit, $\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} V_c(t) \\ I_L(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{C} I_c(t) \\ \frac{1}{L} V_L(t) \end{bmatrix} = 0 \rightarrow \begin{cases} I_c(t) = 0 \\ V_L(t) = 0. \end{cases} \rightarrow$ Open circuit for C, wire for L



$I_R = I_L = I_d$, $V_{in} - I_R R - V_c = 0$, $V_d = V_c$.

$\rightarrow I_L = g(V_c)$, $V_{in} - I_L R - V_c = 0$ (same conclusion!)

Linearization model: $J_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$, $J_u\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}$. $\vec{f}(\vec{x}(t), u(t)) = \begin{bmatrix} \frac{1}{C}(x_2 - g(x_1)) \\ \frac{1}{L}(u - Rx_2 - x_1) \end{bmatrix}$.

$J_{\vec{x}}\vec{f} = \begin{bmatrix} -\frac{g'(x_1)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$, $J_u\vec{f} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$. $\rightarrow A = J_{\vec{x}}\vec{f}(\vec{x}^*, u^*) = \begin{bmatrix} -\frac{g'(x_1^*)}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$.

stability conditions? $tr(A) = \frac{-g'(x_1^*)}{C} - \frac{R}{L}$, $det(A) = \frac{g'(x_1^*)R}{LC} + \frac{1}{LC}$.

for stability, $tr(A) < 0$ & $det(A) > 0$. If $g'(x_1^*) > 0$, A is stable!

Complex Inner Products

Recall Schur Decomposition: For any $A \in \mathbb{R}^{m \times n}$ with real evalues, we can find an orthogonal matrix U s.t. $U^T A U$ is upper-triangular.

Actually, we don't have to make assumptions about evalues being real.

Recall inner product & norm in $\mathbb{R}^n$: $\langle \vec{x}, \vec{y} \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} x_i y_i = \vec{y}^T \vec{x} = \vec{x}^T \vec{y}$.

$\|\vec{x}\|_{\mathbb{R}^n}^2 = \sum_{i=1}^{n} x_i^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle$.

Properties that must be satisfied by a norm in any vector space:

Vector space $\mathbb{V}$ (eg. $\mathbb{R}^n, \mathbb{C}^n$), Field $\mathbb{F}$ ($\mathbb{R}, \mathbb{C}$).

i) $\|\vec{x} + \vec{y}\|_{\mathbb{V}} \leq \|\vec{x}\|_{\mathbb{V}} + \|\vec{y}\|_{\mathbb{V}}$ for any $\vec{x}, \vec{y} \in \mathbb{V}$.

ii) $\|\alpha \vec{x}\|_{\mathbb{V}} = |\alpha| \cdot \|\vec{x}\|_{\mathbb{V}}$ for any $\vec{x} \in \mathbb{V}, \alpha \in \mathbb{F}$.

iii) $\|\vec{x}\|_{\mathbb{V}} \geq 0$ for any $\vec{x} \in \mathbb{V}$ and $\|\vec{x}\|_{\mathbb{V}} = 0 \Leftrightarrow \vec{x} = \vec{0}$.

What is a norm that works for $\mathbb{C}^n$? Not $\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

$\vec{x} := \begin{bmatrix} 1 \\ 2i \end{bmatrix}$. $x_1^2 + x_2^2 = 1 - 4 = -3 < 0 \rightarrow$ doesn't satisfy iii).

Instead, $\|\vec{x}\|_{\mathbb{C}^n}^2 = \sum_{i=1}^{n} |x_i|^2$. Then, $\|\vec{x}\| = \sqrt{|1|^2 + |2i|^2} = \sqrt{1+4} = \sqrt{5}$.

Inner Product: $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} := \sum_{i=1}^{n} x_i \overline{y_i}$. *

- $\langle \vec{x}, \vec{x} \rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} |x_i|^2 = \|\vec{x}\|_{\mathbb{C}^n}^2$.
- $\sum_{i=1}^{n} x_i \overline{y_i} = [\overline{y_1} \cdots \overline{y_n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (\overline{\vec{y}})^T \cdot \vec{x} = \vec{y}^* \vec{x}$ ($\vec{y}^*$ is "conjugate transpose")
- $\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n} = \overline{\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n}}$, so $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = \langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n}$ is not necessarily true!

($\langle \vec{y}, \vec{x} \rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} y_i \overline{x_i}$, $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} x_i \overline{y_i}$)

- Orthogonality: For $\vec{x}, \vec{y} \in \mathbb{C}^n$ is $\vec{x} \perp \vec{y}$ if $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = 0$.

  Orthonormal if $\|\vec{x}\|_{\mathbb{C}^n} = 1$, $\|\vec{y}\|_{\mathbb{C}^n} = 1$ in addition.

  ex) $\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix}$, $\vec{y} = \begin{bmatrix} j \\ 1 \end{bmatrix}$. $\langle \vec{x}, \vec{y} \rangle_{\mathbb{C}^n} = [-j \quad 1] \begin{bmatrix} 1 \\ j \end{bmatrix} = -j + j = 0 \rightarrow \vec{x} \perp \vec{y}$.
- If $Q \in \mathbb{C}^{m \times n}$ has orthonormal columns, then $Q^* Q = I$.

  $\left(\begin{bmatrix} \vec{q}_1^* \\ \vdots \\ \vec{q}_n^* \end{bmatrix} [\vec{q}_1 \cdots \vec{q}_n] = \begin{bmatrix} \vec{q}_1^* \vec{q}_1 & \cdots & \vec{q}_1^* \vec{q}_n \\ \vdots & \ddots & \vdots \\ \vec{q}_n^* \vec{q}_1 & \cdots & \vec{q}_n^* \vec{q}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \right)$

Definition: A square matrix $Q \in \mathbb{C}^{n \times n}$ with orthonormal columns *
is called a "unitary matrix", generalizing orthogonal matrix to $\mathbb{C}$.

$$Q^* Q = I \rightarrow Q^* = Q^{-1} \rightarrow QQ^* = I$$

Schur Revisited: For any $A \in \mathbb{C}^{n \times n}$, we can find a unitary matrix U s.t. $U^* A U$ is upper-triangular. If $A \in \mathbb{R}^{n \times n}$ but its evalues are not real, this generalized version still is valid.

Gram-Schmidt also generalizes to $\mathbb{C}^n$ nicely.