CS 177

Efficiency in Economics

Preliminaries: Let X be a set.

Def) Binary Relation on X is a subset $B \subseteq X \times X$ of the Cartesian product of X with X.

ex) $X := \{\text{Prof}, \text{GSI}, \text{Ugrad}\}$. Binary relation is "older than".

$B = \{(\text{Prof}, \text{GSI}), (\text{GSI}, \text{Ugrad}), (\text{Prof}, \text{Ugrad})\}$.

* B can contain cycles!

Notation: $(x,y) \in B$ is understood as x "B" y. (order sensitive)

We will use binary relations to capture preferences.

Properties of BR:

1) B is complete (or total) if $\forall x,y \in X$, $(x,y) \in B \lor (y,x) \in B$.
2) B is transitive if $\forall x,y,z \in X$, $(x,y) \in B \land (y,z) \in B \Rightarrow (x,z) \in B$.
3) B is antisymmetric if $(x,y) \in B \land (x,y) \in B \Rightarrow x = y$.

Def) A BR that is complete and transitive is called a weak order.

A weak order that is antisymmetric is called a linear order.

In economics, a weak order is called a preference relation.

Also, a linear order is called a strict preference.

ex) Students' preferences of dorm rooms.

Distance to a certain hall is strict (no ties)

Room size if preference (ties are possible)

Notation: Preference Relations are usually denoted by $\succsim$.

A PR $\succsim$ has a utility representation $u: X \to \mathbb{R}$ if u has the property $x \succsim y \iff u(x) \geq u(y)$.

ex) "older than" can be represented by $u(x) := x$'s age.

Given a PR $\succsim$, we may define two derived BRs.

1) The strict part of $\succsim$, denoted $\succ$, is $x \succ y$ if $x \succsim y$ and it is not the case that $y \succsim x$.

2) The indifference relation of $\succsim$, denoted $\sim$, is $x \sim y$ if $x \succsim y \wedge y \succsim x$.

Observation: An indifference curve is the set of indifference relations.

Allocation Problems

ex) Students $:= \{1, 2, 3\}$. Dorms $:= \{\theta_1, \theta_2, \theta_3\}$. Outside $:= \emptyset$.

$X := \text{Dorms} \cup \{\emptyset\}$. Each student i has a PR $\succsim_i$ on X.
(Assume $\succsim_i$ is strict for all i.) Rankings are as follows:

$\succsim_1$	$\succsim_2$	$\succsim_3$
θ_1	θ_1	θ_3
θ_2	θ_3	θ_2
$\boxed{\theta_3}$	$\boxed{\theta_2}$	$\boxed{\theta_1}$
$\emptyset$	$\emptyset$	$\emptyset$

◯ and ◯ are both valid assignments.

How to measure which is "better"?

↳ this is relative to each student.

However, consider □. Can we argue that this is always worse than ◯ or ◯?

Def) Allocation Problem: Tuple $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$ where

1) A is a non-empty finite set of agents.
2) O is a non-empty finite set of objects.
3) $\emptyset$ represents an outside option.
4) $\forall i \in A$, $\succsim_i$ is a strict preference over $O \cup \{\emptyset\}$.

Def) Allocation: function $\mu : A \to O \cup \{\emptyset\}$ s.t. if $\mu(i) = \mu(j)$ when $i \neq j$, it must be that $\mu(i) = \mu(j) = \emptyset$ (repeatable)

Def) Pareto Dominated: μ is PD by μ' if $\forall i \in A$, $\mu'(i) \succsim_i \mu(i)$ and $\exists j \in A$, $\mu'(i) \succ_i \mu(i)$.

Def) Pareto Optimal: μ is PO if it is not PD by any other μ'.

Def) Ordering of Agents: total order $\geq$ on A.

Algorithm) Serial Dictatorship Allocation:

Input: An allocation problem $(A, O, \phi, \{\succsim_i \mid i \in A\})$ and ordering $\geq$ of agents.

Initialize: $\bar{A} \leftarrow A$, $\bar{O} \leftarrow O$.

while $\bar{A}$ is not empty:

  $i \leftarrow$ top agent in $\bar{A}$ for $\geq$.

  $\mu(i) \leftarrow$ top choice according to $\succsim_i$ in $\bar{O} \cup \{\phi\}$.

  $\bar{A} \leftarrow \bar{A} \setminus \{i\}$, $\bar{O} \leftarrow \bar{O} \setminus \{\mu(i)\}$.

Output: μ.

Runtime: $O(nk)$ where $n = |A|$, $k = |O|$.

Theorem) Let $(A, O, \phi, \{\succsim_i \mid i \in A\})$ be an allocation problem. Then, an allocation μ is PO iff it is the output of SDA($\exists \geq$).

Proof: First, we prove that if μ is the output of SDA with ordering $\geq$, μ is PO (backward implication). WLOG, suppose that the ordering $\geq$ is $1 \geq 2 \geq 3 \cdots \geq n-1 \geq n$ for $A = [n]$.

Let μ' be any assignment with $\mu'(i) \succsim_i \mu(i)$ $\forall i \in [n]$. We will prove that no μ' is strictly better than μ for any agent. First, observe that $\mu(1) \succsim_i \mu'(1)$ since $\mu(1)$ is 1's top choice in $O \cup \{\emptyset\}$. Then, since $\succsim_1$ is strict, it must be the case that $\mu(1) = \mu'(1)$. This sets the base case. Now, suppose that $\mu'(i) = \mu(i)$ for all $i \in \{1, \dots, k-1\}$. We need to prove that $\mu'(k) = \mu(k)$. Because μ' is an assignment, $\mu'(k)$ cannot be one of the objects in $\mu(i)$ for $i \in \{1, \dots, k-1\}$ unless it is $\emptyset$. Then $\mu'(k)$ was available to k when it was their time to choose in SDA. Hence, $\mu(k) \succsim_i \mu'(k)$, and since $\succsim_i$ is strict, $\mu(k) = \mu'(k)$. By induction, we conclude that $\mu' = \mu$ and that μ is PO. //

Now, we prove the forward direction: if μ is PO, $\exists \geq$ on A s.t. μ is the outcome of SDA with ordering $\geq$.

Lemma 1: some agent is getting their favorite option in $O \cup \{\emptyset\}$.

$\because$ If some agent i's top choice is $\emptyset$, $\mu(i) = \emptyset$ by PO of μ.

Suppose that no agent gets their top choice in μ. Then, no agent's top choice is $\emptyset$. Let agent i_1 have 1's top choice in μ.

Let agent i_2 have i_1's top choice in μ. Given agent i_{k-1}, let agent i_k have i_{k-1}'s top choice. This defines a sequence $i_1, i_2, \ldots$. Since A is finite, $\exists k$ and S s.t. $i_k \neq i_{k+1} \neq \cdots \neq i_{k+S} = i_k$. The agents $i_k \ldots i_{k+S-1}$ are all distinct. Now, define a new assignment μ' by letting $\mu'(i_\ell) = \mu(i_{\ell+1})$ $\forall \ell \in \{k, \ldots, k+S-1\}$ and let $\mu'(i) = \mu(i)$ for all other agents. Then $\mu'(i_\ell) \succ_i \mu(i_{\ell+1})$ $\forall \ell \in \{k, \ldots, k+S-1\}$ and $\mu'(i) \succsim_i \mu(i)$ for all other agents.
This is absurd, as μ is PO. //

Lemma 2: μ is PO, and $\bar{A} \subseteq A$, $\bar{O} := \{\mu(i) \mid i \in A'\}$, then $\mu|_{A\setminus\bar{A}}$, the restriction of μ to $A \setminus \bar{A}$, is PO in the assignment problem $(A\setminus\bar{A}, O\setminus\bar{O}, \emptyset, \{\succsim_i \mid i \in A\setminus\bar{A}\})$.

$\because$ Suppose towards a contradiction that $\exists$ assignment μ': $A\setminus\bar{A} \to (O\setminus\bar{O}) \cup \{\emptyset\}$ s.t. $\mu'(i) \succsim \mu(i)$ $\forall i \in A\setminus\bar{A}$ and $\mu'(i) \succ_i \mu(i)$ $\exists i \in A\setminus\bar{A}$. Now let $\mu^*: A \to O \cup \{\emptyset\}$ in $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$ by $\mu^*(i) := \begin{cases} \mu'(i) & \text{if } i \in A\setminus\bar{A} \\ \mu(i) & \text{if } i \notin \bar{A} \end{cases}$. Then, $\mu^*(i) \succsim_i \mu(i)$ $\forall i \in A$ while $\mu^*(i) = \mu'(i) \succ_i \mu(i)$ $\exists i \in A\setminus\bar{A} \subseteq A$.
Absurd, since μ is PO. //

$\Rightarrow$ By Lemma 1, $\exists i \in A$ s.t. $\mu(i)$ is top choice of $\succsim_i$. Let this be the first agent. Reasoning by induction, if we already ordered upto $k-1$, $i_1 > i_2 > \dots > i_{k-1}$, by Lemma 2, if $\bar{A} = \{i_1, i_2, \dots, i_{k-1}\}$, then $\mu|_{A \setminus \bar{A}}$ is PO. In $(A \setminus \bar{A}, O \setminus \{\mu(i_1), \dots, \mu(i_{k-1})\}, \emptyset, \{\succsim_i \mid i \in A \setminus \bar{A}\})$. By Lemma 1, $\exists i \in A \setminus \bar{A}$ gets a top choice in the remaining objects. Let i_k be this i. Induction is complete. //

Def) Social Choice Problem: tuple $(A, X, \{\succsim_i \mid i \in A\})$ in which A is a finite nonempty set of agents, X is a nonempty set of outcomes, and $\forall i \in A$, $\succsim_i$ is a preference over X.

ex) An assignment problem $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$ is a SCP in which $X := \{\text{all assignments } \mu\}$, and $\mu \succsim \mu'$ iff $\mu(i) \succsim_i \mu'(i)$ $\forall i$.

Def) An outcome $x \in X$ is PD if $\exists x' \in X'$ s.t. $x' \succsim_i x$ $\forall i \in A$, and $x' \succ_i x$ $\exists i \in A$. An outcome is PO if it is not PD.

Fairness in Economics

The Cake Cutting Problem: Model of infinitely divisible objects

↳ 2 agents has a famous solution → A cuts, B picks first

⇒ intuitively, for n agents, induct from the $(n-1)$ case!

Def) Cake Cutting: tuple $(X, A, \{u_i \mid i \in A\})$ in which:

1) $X := [0,1]$ represents the cake (infinitely divisible resource)
2) $A := [n]$ is the set of agents
3) $\forall i \in A$, $u_i : \mathcal{I} \to \mathbb{R}$ is a utility function, where the domain $\mathcal{I}$ is the set of all finite unions of intervals (e.g. $[0, 1/3] \cup (2/3, 4/5] \in \mathcal{I}$)

Def) Partition (of $[0,1]$): a collection of sets $P_1, \ldots, P_n$ with:

1) $\forall i$, $P_i \in \mathcal{I}$ 2) $\forall i,j$, $P_i \cap P_j = \emptyset$ if $i \neq j$ 3) $\bigcup_{i=1}^{n} P_i = [0,1]$

Assumptions on utility:

1) $\forall i$, $u_i([0,1]) = 1$ and $u_i(\emptyset) = 0$.
2) $u_i(P \cup P') = u_i(P) + u_i(P')$ for $P, P' \in \mathcal{I}$, $P \cap P' = \emptyset$.
3) $\forall \alpha \in (0,1)$, $\exists$ interval $[a,b]$ with $u_i([a,b]) = \alpha$.
4) $u_i(P) \geq 0$ $\forall P \in \mathcal{I}$.

Leading Example: Suppose for all i, $\exists$ function $f_i : [0,1] \to \mathbb{R}_+$ s.t. $u_i([a,b]) = \int_a^b f_i(x)dx$ and $\int_0^1 f_i(x)dx = 1$. (a PDF)

Def) Proportionality: A partition $P_1, \dots, P_n$ s.t. $u_i(P_i) \geq \frac{1}{n} = \frac{u_i([0,1])}{n}$.

Def) Envy-Free: A partition $P_1, \dots, P_n$ s.t. $\forall i, j$, $u_i(P_i) \geq u_i(P_j)$.

Obs) If a partition is envy-free, then it is proportional.

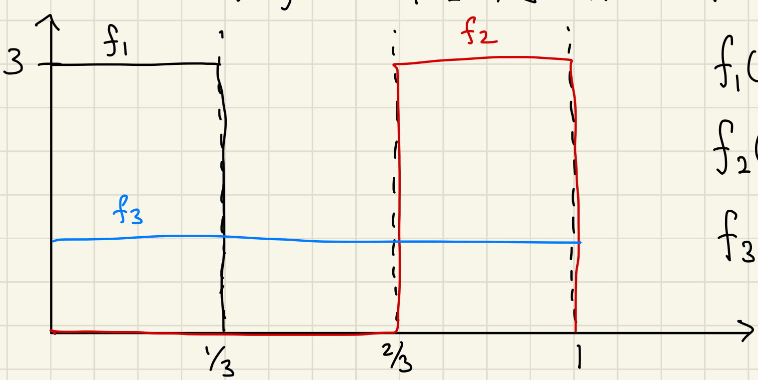
Proof: Let $P_1, \dots, P_n$ be envy-free, then $u_i(P_i) \geq u_i(P_j)$ $\forall j \in [n]$.

$$n \cdot u_i(P_i) = \sum_{j=1}^{n} u_i(P_i) \geq \sum_{j=1}^{n} u_i(P_j) = u_i\left(\bigcup_{j=1}^{n} P_j\right) = u_i(X) = 1.$$

$\Rightarrow u_i(P_i) \geq \frac{1}{n}$, which is proportionality. //

However, a division may be proportional, but not envy-free.

ex) $A = \{1, 2, 3\}$, $u_i := f_i : [0,1] \to \mathbb{R}$ as follows:



$$f_1(x) = \begin{cases} 3 & \text{if } x \in [0, 1/3] \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} 3 & \text{if } x \in [2/3, 1] \\ 0 & \text{otherwise} \end{cases}$$

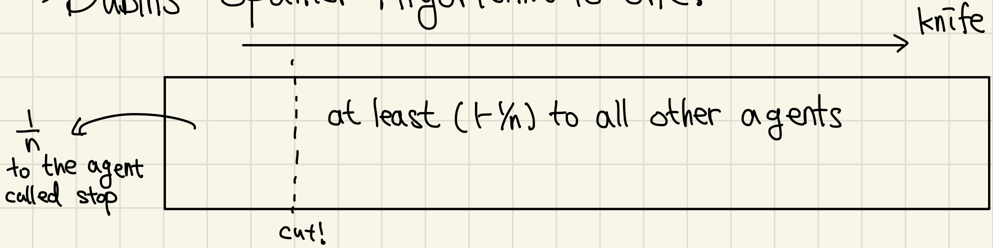
$$f_3(x) = 1.$$

Consider the partition $P_1 = [0, 1/9]$, $P_2 = (1/9, 8/9)$, $P_3 = [8/9, 1]$.

This is proportional. However, agents 1 & 2 will envy 3. //

Is there an efficient algorithm guaranteeing proportionality?

↳ Dubins-Spanier Algorithm is one.



Proof: $n=2$, then obviously proportional since agent who calls stop gets utility $1/2$, and the other gets utility $\geq 1/2$.

Suppose that any problem with $(n-1)$ agents gets a proportional partition. With n agents, the one who calls gets utility $1/n$. For the remaining agents, the remaining cake is worth $\geq (1-\frac{1}{n}) = \frac{n-1}{n}$. By induction hypothesis, the algorithm gives each remaining agent at least $\left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{n-1}\right) = \left(\frac{1}{n}\right)$ of the cake. //

How to analyze complexity of this? → Query Complexity!

Two oracles: 1) $\mathrm{Eval}_i([a,b])$ returns $u_i([a,b])$,

2) $\mathrm{Cut}_i(a,\alpha)$ returns b s.t. $u_i([a,b]) = \alpha$.

Algorithm) Dubins-Spanier:

Initialize: $\bar{A} \leftarrow A$, $\bar{X} \leftarrow [0,1]$.

While $\bar{A}$ is nonempty, do:

1) Let a s.t. $\overline{X} = [a, 1]$.

2) Let $C_i := \mathrm{Cut}_i(a, 1/n)$.

3) Let $i \in \overline{A}$ s.t. $C_i \leq C_j \ \forall j \in \overline{A}$

4) $P_i \leftarrow [a, C_i)$, $\overline{A} \leftarrow \overline{A} \setminus \{i\}$.

Output: Partition $\{P_1, \ldots, P_n \cup \{1\}\}$ of the cake.

Analysis of QC: $O(n^2)$ calls to Cut_i (n loops, $(n \ldots 1)$ calls each)

Claim: We can do better. $\rightarrow$ Use binary search!

Evan-Paz (Assume $n = 2^k$, $k > \emptyset$):

Subroutine(k): Given interval $[a, b]$, let $C_i := \mathrm{Cut}_i(a, 1/2)$. Order agents s.t. $C_1 \leq C_2 \leq \ldots \leq C_n$. Then, create two subproblems;

1) $A = \{1, \ldots, n/2\}$, $X = [a, C_{n/2})$, 2) $A = \{n/2 + 1, \ldots, n\}$, $X = [C_{n/2+1}, b]$.

Call subroutine($k-1$) for each subproblem unless $k = 1$, which is the base case of having a single agent in each subproblem, where we can assign that agent the remaining portion of the cake.

Obs) There could be "middle pieces" that are unassigned. Give those to any agent arbitrarily.

Claim) Evan-Pas is proportional.

Proof) By induction on K. Base Case: $K=1 \to$ agent 1 gets $[a, C_1]$, and agent 2 gets $[C_2, 1]$, which are each worth $\frac{1}{2}$ to them.

Suppose $(K-1)$ case holds. Then, in the subroutine(K), each agent in a subproblem gets (by IH) $\frac{1}{2^{k-1}}$ of the cake in the subproblem. But the cake in the subproblem was worth at least $\frac{1}{2}$ of the whole cake to them. So, they get at least $\frac{1}{2} \cdot \frac{1}{2^{k-1}}$ $= \frac{1}{2^k} = \frac{1}{n}$ of the whole cake. //

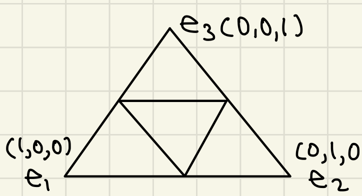
Analysis of QC: $O(n \log n)$ calls to Cut_i ($\log n$ calls for every agent)

What about envy-freeness?

Theorem) There exists an envy-free partition for any cake division problem.

↳ The proof of this theorem relies on Sperner's Lemma.

Digression) Sperner's Lemma: Triangle $T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0 \wedge x_1 + x_2 + x_3 = 1\}$. Vertex$(T) = \{e_1, e_2, e_3\}$.



Def) Triangulation: a collection $T_1, \ldots, T_k$ of triangles s.t. T is the union of $T_1, \ldots, T_k$ and for any two triangles T_i and T_j, $T_i \cap T_j$ is either disjoint, a vertex, or a side (when $i \neq j$).

Def) Sperner Coloring: function c: Vertices$(T_1, \ldots, T_k) \to \{1,2,3\}$ s.t. if X is a convex combination of e_i and e_j, $c(X) \in \{i, j\}$.

Def) Rainbow Triangle: Triangle T_i s.t. coloring $c(\text{vertex}(T_i)) = \{1,2,3\}$.

Lemma) Sperner's Lemma: If c is a Sperner coloring, then at least one (in fact, an odd number of) triangle is a rainbow triangle. (Proof is by a parity argument from 1-D)

Proof (in $n=3$ agents): Fix $m \in \mathbb{N}$ and a triangulation $T_1, \ldots, T_k$ s.t. if $x, y \in T_i$, then $\|x-y\| \leq 1/m$. Assign labels from $\{A,B,C\}$, to the vertices of $T_1, \ldots, T_k$ s.t. each T_i has all three labels. Define a function c from vertices of $T_1, \ldots, T_k$ to $\{1,2,3\}$ by $c(x) = j$ if $u_a(p^j) \geq u_a(p^h)$ for the division $p^1 = [0, x_1)$, $p^2 = [x_1, x_1+x_2)$, $p^3 = [x_1 + x_2, 1]$ and $a \in \{A,B,C\}$ being the label of the vertex x. c is a Sperner coloring because if x is a convex combination of e_i and e_j, the piece chosen by utility u_a must be either i or j.

By Sperner's Lemma, there exists a triangle T_i s.t. if T_i has vertices x_i^1, x_i^2, x_i^3, then $c(x_i^1)=1$, $c(x_i^2)=2$, and $c(x_i^3)=3$. These are owned by A, B, and C, and call these vertices x^m, y^m, z^m, respectively. $\{c(x^m), c(y^m), c(z^m)\}=\{1,2,3\}$. The sequence $(x^1,y^1,z^1),(x^2,y^2,z^2)\ldots$ in T must have a convergent subsequence. Let this subsequence be $(x^{m_\ell}, y^{m_\ell}, z^{m_\ell})$ where $\ell \geq 1$, and let $w^* = \lim_{\ell\to\infty} x^{m_\ell} = \lim_{\ell\to\infty} y^{m_\ell} = \lim_{\ell\to\infty} z^{m_\ell}$. The limit is the same as $\|x^{m_\ell}-y^{m_\ell}\| < \frac{1}{m^\ell} \to 0$ and $\|x^{m_\ell}-z^{m_\ell}\| < \frac{1}{m_\ell} \to 0$. Then there must be some order of the pieces that occurs infinitely often. WLOG, say $(1,2,3)=(c(x^{m_\ell}), c(y^{m_\ell}), c(z^{m_\ell}))$ for infinitely many ℓ. So there is a further subsequence $(x^{m_{\ell_h}}, y^{m_{\ell_h}}, z^{m_{\ell_h}})$ for $h\geq 1$ with $(c(x^{m_{\ell_h}}), c(y^{m_{\ell_h}}), c(z^{m_{\ell_h}}))=(1,2,3)$ for $h \geq 1$. The vector $w^*=(w_1^*, w_2^*, w_3^*)$ is associated with the division $P_1(w^*)$ to A, $P_2(w^*)$ to B, and $P_3(w^*)$ to C. This is envy-free since if, say A envies B, then $U_A(P_1(w^*)) < U_A(P_2(w^*))$, which would imply by continuity of U_A that $U_A(P_1(x^{m_{\ell_h}})) < U_A(P_2(x^{m_{\ell_h}}))$ for a large enough h. This is impossible as $P_1(x^{m_{\ell_h}})$ is a favorite piece from $P(x^{m_{\ell_h}})$ to A. //

Theorem) Stromquist: There is no finite algorithm for finding a simple envy-free division (with $n \geq 3$).

Theorem) Aziz & McKenzie: There is an algorithm that computes an envy-free division with n agents w/ query complexity $O(n^{n^{n^{n^{n^{n}}}}})$.

Fairness in Cost & Value Sharing

Ex) Columbia $\to$ C, Paris II $\to$ P, Oxford $\to$ O.

Travel expenses:

C	P	O	CP	CO	OP	COP
650	1200	1300	1350	1400	1450	1600

Def) Game in Characteristic Function Form (Tranfserrable Utility Game):

A pair (N, v) in which:

1) N is a nonempty finite set of players.

2) $v: \underbrace{2^N}_{\text{power set of } N} \to \mathbb{R}^+$ is the characteristic function of the game.

We assume that $v(\emptyset) = 0$, and $A \subseteq B \Rightarrow v(A) \leq v(B)$.

Def) Coalition: A subset $A \subseteq N$.

Two intepretations of the game (N, v):

1) $\forall A \subseteq N$, $v(A)$ is the total value (utility) that can be generated

by the players in A alone.

2) $\forall A \subseteq N$, $v(A)$ is the cost of serving the players in A with some desireable benefit.

Ex) $N = \{1,2,3\}$. $u(A) = \begin{cases} 1 & \text{if } |A| \geq 2 \\ 0 & \text{otherwise} \end{cases}$ ("divide a dollar by majority")

Ex2) $N = \{1,2,3\}$. $1 \rightarrow$ seller of an indivisible good worth 0 to themself.

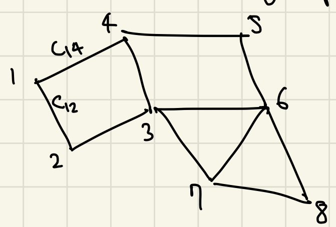
$2 \rightarrow$ buyer who values the good $V_2 > 0$.

$3 \rightarrow$ 〃 $V_3 > V_2 > 0$.

$\Rightarrow v(\{i\}) = 0$ for $i = 1,2,3$. $V(\{2,3\}) = 0$. $V(\{1,2\}) = V_2$.

$V(\{1,3\}) = V_3$. $V(\{1,2,3\}) = V_3$.

Ex3) Cost in a graph: $N = [8]$, $\forall A$, $v(A) :=$ cheapest way to connect all agents in A. (MST for $A = N$)



Def) Marginal Contribution: For a CFG, player i's marginal contribution to coalition $S \subseteq N$ is $\Delta_i^v(S) := V(S \cup \{i\}) - V(S \setminus \{i\})$.

Recall that an ordering of players in N is a complete, transitive, and antisymmetric binary relationship. Denote by Π the set of all ordering of N. ($|\Pi| = n!$ where $n := |N|$)

Def) Shapley Value: A player $i \in N$ in a game (N, v) has SV of $\varphi_i(v) := \frac{1}{n!} \sum_{\geq \in \Pi} \Delta_i^v(S(\geq, i))$ where $S(\geq, i) := \{j \in N \mid j \geq i\}$ is the set of all players who precedes i in the ordering $\geq$.

→ For the Traveling Expenses example, $C = 308.33$, $P = 608.33$, $O = 683.33$

Idea: If $x \in \mathbb{R}^d$, write $x = [x_1, \ldots, x_d]$ so that we can think x as a function from the set $\{1, 2, \ldots, d\}$ into $\mathbb{R}$. The value of the function at h is x_h. For a game (N, v), think of v as a vector in the space $\mathbb{R}^{2^N - 1}$ (there are $2^N - 1$ nonempty coalitions).

Fix a set N of n players. Let Γ be the set of all functions $v : 2^N \to \mathbb{R}^+$ s.t. (N, v) is a CFG.

Def) Solution: function $s : \Gamma \to \mathbb{R}_+^N$ s.t. $\forall v \in \Gamma$, $\sum_{i=1}^{n} \overbrace{s_i(v)}^{S(v)_i} = v(N)$.

($S(v) \in \mathbb{R}_+^N$, so $S(v) = [S_1(v), \ldots, S_n(v)]$)

Def) Substitutes: players i and j s.t. $\forall$ game v, $\forall A \subseteq N$, $A \not\ni i$, $A \not\ni j$, $\Delta_i^v(A) = \Delta_j^v(A)$.

Axioms: 1) If $\Delta_i^v(A) \geq \Delta_i^w(A)$ $\forall A \subseteq N$, then $S_i(v) \geq S_i(w)$. (marginality)

2) If i and j are substitutes in game v, then $S_i(v) = S_j(v)$. (substitute players)

Theorem: [Young] A solution satisfies the marginality and substitute players axiom iff it is the Shapley Value.

Each $v \in \Gamma$ is a function $v: 2^N \to \mathbb{R}^+$ with $v(\emptyset) = 0$. So, it is a vector in $\mathbb{R}^{2^N - 1}$.

Def) Simple Games: a coalition $T \subseteq N$, $T \neq \emptyset$, $V_T(A) = \mathbb{1}\{T \subseteq A\}$.

Lemma) The collection of all simple games form a basis for $\mathbb{R}^{2^N-1}$.

Proof: Let's prove that simple games are linearly independent. Suppose towards a contradiction that there exists numbers $a_T \in \mathbb{R}$ for all $T \in N$, $T \neq \emptyset$ s.t $\vec{0} = \sum_{T \subseteq N, T \neq \emptyset} a_T V_T$ and at least one a_T is nonzero. Let T^* be a coalition with $a_{T^*} \neq 0$ and minimal w.r.t. this property, i.e. $T \subsetneq T^*$ $\Rightarrow a_T = 0$. Then, $\vec{0} = \sum_{T \subseteq N, T \neq \emptyset} a_T V_T(T^*) = \sum_{T \subseteq T^*} a_T V_T(T^*) + \sum_{T \nsubseteq T^*, T \neq \emptyset} a_T \cancelto{0}{V_T(T^*)}$

$\Rightarrow \vec{0} = a_{T^*} \underbrace{V_{T^*}(T^*)}_{1}$. Contradiction since $a_{T^*} \neq 0$. //

Lemma) If s satisfies marginality and $\Delta_i^v(A) = \Delta_i^w(A)$ $\forall A \subseteq N$, then $S_i(v) = S_i(w)$. ($\because S_i(v) \geq S_i(w) \wedge S_i(v) \leq S_i(w)$.)

Proof of Young's Theorem: Recall that the trivial game $\vec{0}\in\Gamma$ has $S_i(\vec{0})\geq 0$ for all i and $\sum_{i=1}^{n} S_i(\vec{0})=0 \Rightarrow \forall i,\ S_i(\vec{0})=0$. By the first lemma, for any $v\in\Gamma$, $\exists\, a_T\in\mathbb{R}$, $T\in P(N):=$ set of all nonempty subsets of N s.t. $v=\sum_{T\in P(N)} a_T v_T$. Let $\mathcal{T}(v):=\{T\in P(N) \mid a_T\neq 0\}$. The proof is by induction on the cardinality of $\mathcal{T}(v)$.

Base Case: $|\mathcal{T}(v)|=1$, so $\exists$ some $T\in P(N)$ with $v=a_T v_T$. Consider first $i\notin T$. Then $\Delta_i^v(A)=0$ for any coalition A. But then, $\Delta_i^v(A)=\Delta_i^{\vec{0}}(A)\ \forall A$. By the second lemma, $S_i(v)=S_i(\vec{0})=0$.

Consider, second, $i,j\in T$. Then i and j are substitute players. By the substitute players axiom, $S_i(v)=S_j(v)=\gamma$. Then,

$$\alpha_T=v(N)=\cancelto{0}{\sum_{i\notin T}S_i(v)}+\sum_{i\in T}S_i(v)=\gamma\cdot|T|\Rightarrow\gamma=\frac{\alpha_T}{|T|}.$$

Hence,

$$S_i(v)=\begin{cases}0 & \text{if } i\notin T\\ \frac{\alpha_T}{|T|} & \text{if } i\in T.\end{cases}$$

(HW: SV satisfies the axioms $\Rightarrow S(v)=\varphi(v)$)

Inductive Step: Suppose that $\forall v\in\Gamma$ with $|\mathcal{T}(v)|\leq k-1$, we have $S(v)=\varphi(v)$.

Consider a game $v\in\Gamma$ with $|\mathcal{T}(v)|=k$. We shall prove that $S(v)=\varphi(v)$.

Let $T^*:=\bigcap_{T\in\mathcal{T}(v)} T$. Recall $v=\sum_{T\in\mathcal{T}(v)} a_T v_T$.

Case 1) $i\notin T^*$. Define $w:=\sum_{T\in\mathcal{T}(v),\, T\ni i} a_T v_T$, then $|\mathcal{T}(w)|<|\mathcal{T}(v)|=k$, so $S_i(w)=\varphi_i(w)$.

But for any coalition A, $\Delta_i^v(A)=v(A\cup\{i\})-v(A\setminus\{i\})=$

$$\sum_{T\in\mathcal{T}(v)} a_T v_T(A\cup\{i\}) - \sum_{T\in\mathcal{T}(v)} a_T v_T(A\setminus\{i\}) = \sum_{T\in\mathcal{T}(v)} a_T \Delta_i^{v_T}(A) = \sum_{T\in\mathcal{T}(v), T\ni i} a_T \Delta_i^{v_T}(A) = \Delta_i^{w}(A)$$

$\hookrightarrow$ by the first lemma, $S_i(v) = S_i(w) = \varphi_i(v)$.

case 2) $i \in T^*$. First, if $T^* = \{i\}$, then $S_i(v) = \varphi_i(v)$ because $S_i(v) = v(N) - \sum_{j\neq i} S_j(v) = v(N) - \sum_{j\neq i}\varphi_j(v) = \varphi_i(v)$. Second, suppose T^* contains at least two players. If $i,j\in T^*$, then $i,j\in T$ $\forall T\in\mathcal{T}(v)$. Hence i and j are substitute players in v_T, and thus they are substitutes in v since $\forall A$, $\Delta_i^v(A) = \sum_{T\in\mathcal{T}(v)} a_T \Delta_i^{v_T}(A)$. Then, by the substitute players axiom, $S_i(v) = S_j(v) := \gamma$. So $\forall i\in T^*$, $S_i(v) = \gamma$. But φ also satisfies the axioms, so there is $\alpha = \varphi_i(v)$ $\forall i \in T^*$. Finally, $\sum_{i\in N} S_i(v) = \sum_{i\notin T^*} S_i(v) + \sum_{i\in T^*} S_i(v) = \sum_{i\notin T^*}\varphi_i(v) + |T^*|\gamma = v(N)$.

$\Rightarrow \gamma\cdot|T^*| = v(N) - \sum_{i\notin T^*}\varphi_i(v) = \sum_{i\in T^*}\varphi_i(v) = \alpha\cdot|T^*|$. Thus, $\gamma = \alpha$, which implies that $S_i(v) = \varphi_i(v)$ $\forall i\in T^*$. //

Application) Interpretable AI: SHAP. Given a set N of features (attributes). Say $|N| = n$. The model will be $f: X\subseteq\mathbb{R}^n \to \mathbb{R}$ (e.g. Neural Networks). For any subset of attributes $S\subseteq N$, let $v(S) := E[f(\tilde{x}) \mid \tilde{x}_i = x_i\ \forall i\in S]$. Then, $\varphi_i(v)$ gives a decomposition of $v(N) = f(x)$, i.e. of the prediction that can be attributed to the feature i.

Application) Voting Games: Set $N := [n]$ of voters. Voter i has w_i votes to cast. q is the supermajority threshold. A game is defined by: $v(S) := \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$. Then $\varphi_i(v)$ computes the "power" of the player i. Ex: $N = \{\text{Mom, Alice, Bob, Carol}\}$, $w = \{20, 5, 2, 1\}$. $q = 18$.

$\rightarrow$ Here, Mom has all of the voting power. Ex2: UN Security Counsel, where $N = 5$ permanent & 10 rotating members, $q = 9$, but permanent members have veto power. $\varphi_{\text{Permanent}} \simeq 19.63\%$, $\varphi_{\text{rotating}} \simeq 0.19\%$.

Application) Claims Problems: "Two hold a garment, both claim it all. Then the one is awarded half, the other half. Two hold a garment, one claims it all, the other claims half. Then the one is awarded $3/4$, the other $1/4$."

Def) Claims Problem: a tuple $(c_1, \ldots, c_n; x)$ of n agents where each agent i has a claim $c_i \geq 0$, and $x \leq \sum_i c_i$ is the total to be shared.

ex) A company goes bankrupt. There are n investors/workers with claims, but assets are worth $x \leq \sum_i c_i$.

ex) A person is deceased. n inheritors. Each was promised an amout $c_i \geq 0$, but when assets were liquidated, they amount to $x \leq \sum_i c_i$.

ex) Suppose Alice & Bob inherit \$300,000. Alice was promised \$200,000, and Bob \$300,000. How much should each get?

↳ According to proportionality, A = \$120,000, B = \$180,000.

↳ According to contested garment, A = \$100,000, B = \$200,000.

→ This is actually the Shapley Value!

Def) Contested Garment: given a claims problem $(c_1, c_2; X)$ with 2 agents, let $m_1 := \max\{X - c_1, \emptyset\}$, and $m_2 := \max\{X - c_2, \emptyset\}$. The solution is to give 1 $S_1 = m_2 + \frac{X - m_1 - m_2}{2}$ and 2 $S_2 = m_1 + \frac{X - m_1 - m_2}{2}$.

ex) Alice: 80, Bob: 60, X = 100. → $S_A = 40 + 20 = 60$, $S_B = 20 + 20 = 40$.

A Problematic Example: 4 agents, Alice, Bob, Chana, Dalia, X = 600.

X	A	B	C	D
600	200	300	200	300
φ	116.66	183.33	116.66	183.33

→ Alice and Bob splits 300 differently from the example above! → inconsistent

Efficiency and Fairness

Recall: An assignment problem is a tuple $(O, \emptyset, A, \{\succsim_i \mid i \in A\})$ where O is a finite set of objects, $\emptyset$ is the outside choice, A is a finite set of agents, and $\succsim_i$ is a strict preference on $O \cup \{\emptyset\}$ $\forall i \in A$.

Assume (mainly for convenience) that:

1) $|A| = |O| = n$. 2) ϕ is ranked last by $\forall i \in A$. $\Rightarrow$ All agents can get an object, and we can assume WLOG ignore the outside option.

$\hookrightarrow$ In this case an assignment is a function $\mu : A \to O$ s.t. if $a \neq a'$, then $\mu(a) \neq \mu(a')$ (bijection actually)

Ex) $A = \{a_1, a_2, a_3, a_4\}$, $O = \{o_1, o_2, o_3, o_4\}$.

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
o_1	o_1	o_3	o_4
o_2	o_3	o_1	o_3
o_3	o_2	o_2	o_2
o_4	o_4	o_4	o_1

$\to$ possible PO assignments:

a_1	o_2	o_1	o_1
a_2	o_1	o_2	o_3
a_3	o_3	o_3	o_2
a_4	o_4	o_4	o_4
	231, 213, 321	132, 312	123

Let X be a real $n \times n$ matrix with one row for each agent and one column for each object. Let $X_{a,o}$ be the probability that $a \in A$ gets o.

$$X = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/3 & 1/6 & 0 \\ 0 & 1/6 & 5/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for the example above. In fact, the matrix and random assignments are bijective.

Let $(O, \emptyset, A, \{\succsim_i \mid i \in A\})$ be an assignment problem under our assumptions ($|O| = |A| = n$, $\emptyset$ ranked last) Then,

Def) Random Assignment: an $n \times n$ matrix, with $X_{a,o} \in [0,1]$ $\forall a \in A$, $o \in O$ s.t. 1) $\sum_{o \in O} X_{ao} = 1$ $\forall a \in A$, 2) $\sum_{a \in A} X_{ao} = 1$ $\forall o \in O$.

↳ a RA is fractional if $X_{ao} \in (0,1)$ for some $a \in A, o \in O$, and is integral if $X_{ao} = \{0,1\}$ $\forall a \in A, o \in O$. An integral RA is also called a permutation matrix. In an abuse of terminology, we call them just assignments. → RA is actually a "layering" of weighted assignments!

Obs) If X is a RA, then each row X_a is a vector $\in \mathbb{R}^n_+$ with $\sum_{o \in O} X_{ao} = 1$, so it's a lottery over O.

Theorem) Birkhoff-von Neumann: If x is a RA, then $\exists$ a collection $x^1, \dots, x^K$ of assignments, and numbers $\lambda^1, \dots, \lambda^K$ s.t.:

1) $\lambda^k \geq 0$ $\forall k = [K]$, 2) $\sum_{k=1}^{K} \lambda^k = 1$, 3) $x = \sum_{k=1}^{K} \lambda^k x^k$.

Idea: Round the RA until we reach an assignment.

Def) Alternating Cycle: sequence of entries in the RA matrix X, $X_{a_1 o_1}, X_{a_2 o_2}, \dots, X_{a_M o_M} \in (0,1)$ s.t. 1) all $(a_1 o_1), \dots, (a_M o_M)$ are distinct,

2) If m is odd, $X_{a_m o_m}$ and $X_{a_{m+1} o_{m+1}}$ are on the same row ($a_m = a_{m+1}$), and

if m is even, they are on the same column ($o_m = o_{m+1}$)

3) $o_M = o_1$ (last and first are in the same column)

Lemma) If X is a fractional RA, then it has an alternating cycle.

Proof: First, since X is fractional, we can choose a fractional entry $X_{a_1 o_1}$.

Define $X_{a_m o_m}$ by induction. If m is odd, we know, since $\sum_o X_{a_m o} = 1$, that $\exists$ some $X_{a_m o'}$ that is fractional. Define $X_{a_{m+1} o_{m+1}} \leftarrow X_{a_m o'}$. If m is even, there is $X_{a' o_m} \in (0,1)$. If $a' o_m = a_i o_i$ s.t. $i < m$, stop and relabel $i \leftarrow 1, \dots, m \leftarrow M$. Otherwise, let $X_{a_{m+1} o_{m+1}} \leftarrow X_{a' o_m}$.

The cycle must close since the entries in X is finite. //

Rounding Algorithm: fractional RA $X \longrightarrow$ assignment $\overline{X}$.

while x is fractional, do:

1) Find an alternating cycle (by construction in the lemma)

2) Find the largest ε (can be calculated explicitly) s.t. if we replace $X_{a_m o_m} \leftarrow X_{a_m o_m} + \varepsilon$ for odd m and $X_{a_m o_m} \leftarrow X_{a_m o_m} - \varepsilon$ for even m, then $X_{a_m o_m} \in [0,1]$ ($\Rightarrow \exists X_{a_m o_m} \in \{0,1\}$)

Obs) By the lemma, the algorithm is well defined. After each iteration, x remains a RA, and by our choice of ε, there is one (or more) fewer fractional entries in X.

This means that the algorithm terminates in $\Theta(n^2)$ steps and outputs an assignment $\overline{X}$ with the property that if $\overline{X}_{ao}=1$, then $X_{ao}>0$.

Obs) If X is an $n\times n$ matrix s.t. 1) $X_{ao}\geq 0$ $\forall (a,o)\in A\times O$, for some $c>0$ 2) $\sum_{o\in O} X_{ao}=c$ $\forall a\in A$, 3) $\sum_{a\in A} X_{ao}=c$ $\forall o\in O$, then $\frac{1}{c}X$ is a RA. So, by the lemma, $\exists$ an assignment $\overline{X}$ s.t. if $\overline{X}_{ao}=1 \Rightarrow \frac{1}{c}X_{ao}>0 \Rightarrow X_{ao}>0$, i.e. the lemma applies to any matrix with ≥ 0 entries and constant rows and columns sums.

Proof of BvN Theorem: Apply such algorithm to X to construct $\mathcal{X}$ and Λ.

Input: RA X. Initialize $\mathcal{X}\leftarrow\emptyset$, $\Lambda=\emptyset$.

While $X\neq 0$, do:

1) By the rounding algorithm, find an assignment $\overline{X}$ s.t. $\overline{X}_{ao}=1 \Rightarrow X_{ao}>0$.

2) Find min $X_{ao}>0$ s.t. $\overline{X_{ao}}=1$. Let this value be λ.

3) $\mathcal{X}\leftarrow\mathcal{X}\cup\{\overline{X}\}$, $\Lambda\leftarrow\Lambda\cup\{\lambda\}$.

4) $X\leftarrow X-\lambda\overline{X}$.

Output: $\mathcal{X}$ and Λ.

↳ The algorithm proves the theorem since $X=\sum_{k=1}^{K}\lambda_k X_k$ where

$\Lambda = \{\lambda_1, \dots, \lambda_K\}$ and $X = \{x_1, \dots, x_k\}$ with order preserved. //

Now, consider adding back an outside option $\emptyset$.

Ex) $A = \{a_1, a_2, a_3, a_4\}$, $O = \{o_1, o_2\}$. Preferences are such:

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
o_1	o_2	o_1	o_2
o_2	o_1	o_2	o_1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

X^{RSD}	o_1	o_2
a_1	$5/12$	$1/12$
a_2	$1/12$	$5/12$
a_3	$5/12$	$1/2$
a_4	$1/12$	$5/12$

Randomized SDA

Consider RSD. $\Pr[a_1 \text{ getting } o_1] = \Pr[a_1 \text{ is first}] + \Pr[a_2 \text{ or } a_4 \text{ is first}] \cdot \Pr[a_1 \text{ goes second} \mid a_2 \text{ or } a_4 \text{ is first}]$

$= \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \underline{\frac{5}{12}}$.

$\Pr[a_1 \text{ getting } o_2] = \Pr[a_3 \text{ is first}] \cdot \Pr[a_1 \text{ is second} \mid a_3 \text{ is first}]$

$= \frac{1}{4} \cdot \frac{1}{3} = \underline{\frac{1}{12}}$. Also, $\Pr[a_1 \text{ getting } \emptyset] = 1/2$ by symmetry.

Then, we can fill out all other rows by symmetry.

Obs) RA could be found by setting two extra columns that sum up to 1, corresponding to $\emptyset$.

So, a 4×2 matrix is sufficient representation.

However, if we "trade" $1/12$ chance of one agent to another with $5/12$, we obtain $\bar{X} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, which is more efficient for all agents!

Digression: Let $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$ be an assignment problem. Define a probability $p \in \Delta(O \cup \{\emptyset\})$ as a lottery. So if we denote $O^* := O \cup \{\emptyset\}$, then a lottery specifies a probability $p(\theta) \geq 0 \ \forall \theta \in O^*$ and $\sum_{\theta \in O^*} p(\theta) = 1$.

Given a preference $\succsim$ on O^*, denoted by $U_{\succsim}(\theta) := \{\tilde{\theta} \in O^* \mid \tilde{\theta} \succsim \theta\}$ the <u>upper contour set</u> of $\succsim$ at θ. We say that a lottery p <u>first order stochastically dominates</u> lottery q for $\succsim$ (denoted $p \text{ FOSD}_{\succsim} q$) if $\forall \theta \in O^*$, $p(U_{\succsim}(\theta)) \geq q(U_{\succsim}(\theta))$. Also, we say that p <u>strictly FOSD</u> q if p FOSD q and at least one equality is strict.

Obs) p strictly FOSD q if $p \text{ FOSD}_{\succsim} q$ and $p \neq q$. (proof as exercise)

Def) Let $\succsim$ be a PR over O^* and $v: O^* \to \mathbb{R}$ be a function. v <u>represents</u> $\succsim$ if $v(\theta) \geq v(\theta') \iff \theta \succsim \theta' \ \forall \theta, \theta' \in O^*$.

↳ Given such a function v, we may calculate its expected value:

$$E_p v = \sum_{\theta \in O^*} p(\theta) v(\theta) \text{ under lottery } p.$$

Lemma) $p \text{ FOSD}_{\succsim} q$ iff $E_p v \geq E_q v \ \ \forall v$ that represents $\succsim$ (proof in PSet 2)

↳ $\succsim$ is agnostic to cardinal values of v, as long as they are consistent!

Def) Ordinal Pareto Dominance: a random assignment x OPD another RA x' if $\forall i \in A$, the lottery $x_i \in \Delta(O^*)$ $\text{FOSD}_{\succsim_i}$ x'_i, and it strictly $\text{FOSD}_{\succsim_i}$ x'_i for at least one $i \in A$.

Def) Ordinal Efficiency (Pareto Optimal): a RA x that is not ordinally PD by any other RA x'.

↳ In the previous example, x^{RSD} is not OE since it is OPD by $\overline{x}$.

How do we find such ordinally efficient RA?

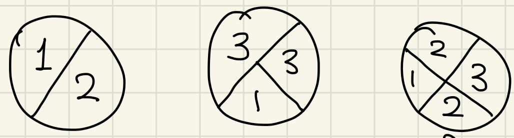
Algorithm) Probabilistic Serial (informally):

ex) $A = \{a_1, a_2, a_3\}$, $O = \{o_1, o_2, o_3\}$, $\succsim :=$

$\succsim_1$	$\succsim_2$	$\succsim_3$
o_1	o_1	o_2
o_2	o_3	o_3
o_3	o_2	o_1

Think of each agent having a "Pac-man" that starts eating away their favorite choice until exhaustion, then move on to the next one.

t	1	2	3
½	o_1	o_1	o_2
¼	o_2	o_3	o_3
¼	o_3	o_3	o_1



o_1 o_2 o_3

Outcome: $$x^{PS} = \begin{array}{cc} & o \\ A & \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}.$$

Back to the first example,

	1	2	3	4
½	o_1	o_2	o_1	o_2

and we are done.

The outcome is exactly $x^{PS} = \overline{x}$.

Claim) x^{PS} will be OE and Envy-Free.

Algorithm) Probabilistic Serial: Let $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$ be an assignment problem. For any subset $O' \subseteq O$, let $\underbrace{M(O', o)}$ be the set of agents $i \in A$ that rank o at the top of O', i.e. $\{i \in A \mid o \succsim_i \tilde{o} \ \forall \tilde{o} \in O'\}$.

Input: Assignment problem $(A, O, \emptyset, \{\succsim_i \mid i \in A\})$.

Initialize: $X \leftarrow 0_{A \times m}$, $A' \leftarrow A$, $O' \leftarrow O$.

while $|A'| > 0$ and $|O'| > 0$:

1) Let $y \in [0,1]$ be the largest number s.t. $\forall o \in O'$,

cake eaten so far

$$\underbrace{\sum_{i \in A} X_{i,o} + y \cdot |M(O', o)| \leq 1}.$$

\# of agents eating o at this iteration

2) $X_{i,o} \leftarrow X_{i,o} + y \quad \forall i \in M(O', o) \ \forall o \in O'$.

3) By def. of y, at least one $o \in O'$ satisfies $\sum_{i \in A} X_{i,o} = 1$. Remove all such objects from O'.

4) Remove from A' all agents who <u>rank $\emptyset$ above any object in O'</u>.

no contest

Output: X. Runtime: $O(|O|)$ iterations.

Ex) (formally) $A = \{a_1, a_2, a_3\}$, $O = \{o_1, o_2, o_3\}$ $\succsim :=$

$\succsim_1$	$\succsim_2$	$\succsim_3$
o_1	o_1	o_2
o_2	o_3	o_3
o_3	o_2	o_1

.

$$X^{RSD} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 5/6 & 1/6 \end{bmatrix} \quad X^{PS} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 3/4 & 1/4 \end{bmatrix}.$$

Def) Envy-Free: RA x that $\forall i \in A$ $x_i \, FOSD_{\succsim_i} \, x_j$ $\forall j \neq i$.

Def) Weak Envy-Free: RA x s.t. $\nexists i,j \in A$ $x_j \succ_i x_i$.

Obs) x^{RSD} is not envy-free. Consider the previous example. $U_{\succsim_1}(O_1) = \{O_1\}$, $U_{\succsim_1}(O_2) = \{O_1, O_2\}$, $U_{\succsim_1}(O_3) = \{O_1, O_2, O_3\}$. Then, we can write a table:

	$Pr \sim x_1^{RSD}$		$Pr \sim x_3^{RSD}$
$U_{\succsim_1}(O_1)$	$1/2$	$>$	0
$U_{\succsim_1}(O_2)$	$4/6$	$<$	$5/6$
$U_{\succsim_1}(O_3)$	1	$=$	1

so neither lottery FOSD for $\succsim_1$.

Def) Ex-post PO: RA x s.t. $\exists$ PO assignments $X_1, X_2, \dots, X_k$ and numbers $\lambda_1, \dots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$ s.t. $x = \sum_{i=1}^{k} \lambda_i X_i$.

Obs) x^{RSD} is ex-post PO (by definition).

Theorem) Let $(A, O, \phi, \{\succsim_i \mid i \in A\})$ be an assignment problem. Let x^{RSD} be the output of the RSD algorithm, and x^{PS} the output of the PS algorithm. Then, 1) x^{RSD} is ex-post PO and weakly envy-free but may not be OE nor EF. 2) x^{PS} is OE and EF.

Given a random allocation x, define a binary relation T_x on O.

Say that $o \, T_x \, o'$ if $\exists$ some agent i for which $o' \succ_i o$ while $x_{i,o} > 0$.

Def) T_x is <u>acyclic</u> if there is no sequence of distict objects $o^1, o^2, \dots, o^k$ s.t. $o^1 T_x o^2, o^2 T_x o^3, \dots, o^k T_x o^1$.

ex) In the $1/12$ example, we had $\begin{bmatrix} 1/12 & 5/12 \\ 5/12 & 1/12 \\ 1/12 & 5/12 \\ 5/12 & 1/12 \end{bmatrix}$. $o_2 T_x o_1$ because a_1 has $x_{1,o_2} = 1/12 > 0$ while $o_1 \succ_1 o_2$ and a_2 has $x_{2,o_1} = 1/12 > 0$ and $o_2 \succ_2 o_1$.

Lemma) A random allocation x is OE iff T_x is acyclic.

Proof: $\Rightarrow$) Suppose that x is not acyclic. We should prove that x is not OE. Since T_x is not acyclic, $\exists$ a sequence of distinct objects s.t. $o^1 T_x o^2 T_x^3 \dots T_x o^1$. Now, $o^k T_x o^{k+1}$ means that $\exists$ some agent i_k s.t. $x_{i_k o_k} > 0$ and $o^{k+1} \succ_{i_k} o^k$. Define x' by having $x'_{i,o} = x_{i,o}$ except for $x'_{i_k,o_k} = x_{i_k,o_k} - \varepsilon$ and $x'_{i_k,o_{k+1}} = x_{i_k,o_{k+1}} + \varepsilon$. Then, $\forall i_k$, $\sum_o x'_{i_k,o} = 1$ since we are adding and subtracting ε same # of times. Similarly, for each object o_k, $\sum_{i \in A} x'_{i,o_k} = 1$ because we $\pm \varepsilon$ once each. Finally, since $x_{i_k,o_k} > 0$, $x'_{i_k,o_k} \geq 0$ for some small enough ε, and $x'_{i_k,o_{k+1}} \leq 1$ since $x_{i_{k+1},o_{k+1}} > 0$ implies that $x_{i_k,o_{k+1}} < 1$. So, $\varepsilon > 0$ small enough $x'_{i_k,o_{k+1}} = x_{i_k,o_{k+1}} \leq 1$. $x' \neq x$, and x'_i FOSD$_{\succsim_i}$ x_i $\forall i \in A$. $\rightarrow$ x is not OE. //

$\Leftarrow$) Suppose X is not OE. Let's show that T_X is not acyclic. Let $X' \neq X$ be a RA where X'_i FOSD$_{\succsim_i}$ X_i $\forall i \in A$. So there must exist some agent i and two objects o and o' s.t. $X_{i,o} > X'_{i,o} \geq \emptyset$, $X'_{i,o'} > X_{i,o'} \geq \emptyset$ and $o' \succ_i o$. Observe then, that $o' \, T_X \, o$. Moreover, since $X'_{i,o'} > X_{i,o'}$, there must exist some agent j s.t. $X_{j,o'} > X'_{j,o'} \geq \emptyset$. But because X'_j FOSD$_{\succsim_j}$ X_j, there must exist some object o'' s.t. $o'' \succ_j o'$ and $X'_{j,o''} > X_{j,o'} \geq \emptyset$. So we conclude that 1) $\exists\, o, o'$ s.t. $o \, T_X \, o'$, 2) whenever $o \, T_X \, o'$, $\exists\, o''$ s.t $o' \, T_X \, o''$. Thus, since O is finite, $\exists$ a sequence of distinct objects with $o^1 \, T_X \, o^2 \, T_X \, o^3 \cdots o^k \, T \, o^1$. //

Lemma) X^{PS} is OE.

Proof: Suppose, towards a contradiction, that X^{PS} is not OE. By the previous lemma, then $\exists\, o^1, o^2, \ldots, o^k$ distinct objects s.t. $o^1 \, T_{X^{PS}} \, o^2$, $o^2 \, T_{X^{PS}} \, o^3$, $\ldots$, $o^{k-1} \, T_{X^{PS}} \, o^k$, $o^k \, T_{X^{PS}} \, o^1$. For each comparison $o^k \, T_{X^{PS}} \, o^{k+1}$ (mod K), $\exists$ an agent i^k s.t. $X^{PS}_{i^k, o^k} > \emptyset$ while $o^{k+1} \succ_i o^k$. Denote by t^k the first time that i^k's pacman starts eating o^k in PSA. Observe that at time t^k, o^{k+1} must have been exhausted. So, t^k must come strictly after t^{k+1} ($t^k > t^{k+1}$). So, $t^1 > t^2$

$> \cdots > t^k > t'$, which is absurd. Contradiction, X^{PS} is OE. //

Proof Idea for EF of X^{PS}: observe that the rate of eating is the same for every agent. So, for each agent, nobody eats more of one's specific preferred ordering (identical if same $\succ_i$).

Social Choice

Def) Social Choice: tuple $(X, A, \{\succsim_i \mid i \in A\})$ in which:

1) X is a nonempty set of outcomes. 2) A is a set of agents.

3) $\forall i \in A$, $\succsim_i$ is a preference relation over X.

Obs) Preferences may not be strict. Also, we usually assume that X and A are finite.

$\succsim_1$	$\succsim_2$	$\succsim_3$
x	y	z
y	z	x
z	x	y

Ex) $X = \{x, y, z\}$, $A = \{a_1, a_2, a_3\}$.

Vote between x and $y \rightarrow x:(1,3), y:(2)$. x vs $z \rightarrow x:(1), z:(2,3)$.

y vs $z \rightarrow y:(1,2), z:(3)$. Majority vote has $x > y > z > x$???

$\Rightarrow$ Condorcet Cycle!

Ex2) Judgement Aggregation (not really social choice)

Contract law: If obligated contractually and failed to comply ⇒ liable.

Three Judges: Alice, Bob, and Judy.

	Obliged	Failed	Liable
A	Y	Y	Y
B	N	Y	N
J	Y	N	N
Maj	Y	Y	N ⇒ ???

Ex3) $X = \{a, b, c, d\}$, $|A| = 21$ with four kinds of preferences:

(3) $\succsim^*$	(5) $\bar{\succsim}$	(7) $\tilde{\succsim}$	(6) $\hat{\succsim}$
a	a	b	c
b	c	d	b
c	b	c	d
d	d	a	a

Suppose everyone votes their top choice. Tally adds up to $\boxed{a=8}$, $b=7$, $c=6$, $d=0$.

↳ This is bad because 13 ppl regards a as the worst choice!

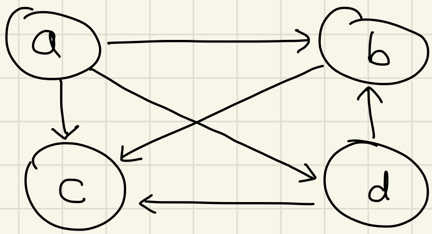
⇒ Borda Count (Score):

	a	b	c	d
$*$	3	2	1	0
$-$	3	1	2	0
$\sim$	0	3	1	2
$\wedge$	0	2	3	1

Tally adds up to ←

$a=24$, $\boxed{b=44}$, $c=38$, $d=20$. $(b > c > a > d)$

Consider a graph s.t. set of vertices $\simeq$ set of outcomes. Edge $x \to y$ exists if a majority prefers y over x.



Based on this graph, c should be selected!
(c is a <u>Condorcet Winner</u>)

Scoring Rules: Suppose $|X| = K$. Fix numbers $S_1 \leq S_2 \leq \ldots \leq S_K$. Assign outcome x S_k points each time one agent ranks x the top k-th position (K is top position, 1 is last). Then, e.g., Borda is $S_k = k-1$, Plurality is $S_k = \mathbb{1}\{k=K\}$.

Ex) Women's Pursuit 2014/15 IBU Biathlon W.C. 7 races, each athlete gets S_k points for placing in the top-k position in one race. Total points are added up. Scores are allocated as such: 1, 2, ..., 31, 32, 34, 36, 38, 40, 43, 48, 54, 60.

Athletes	Total Points
Makrainev	378
Domracheva	377
⋮	⋮
Glazyvina	190

$\xrightarrow[\text{Removed}]{\text{Glazyvina}}$

Athletes	Total Points
Makrainev	378
Domracheva	378 (Tiebreaking win)
~~ ~~	~~ ~~

Recall that $(X, A, \{\succsim_i \mid i \in A\})$ is a social choice problem.

If $A = \{1, 2, \ldots, n\}$, we can write preferences of the agents as a tuple $\succsim = (\succsim_1, \succsim_2, \ldots, \succsim_n)$, which is called a <u>preference profile</u>.

Fix X, a finite set of outcomes. Fix $A = [n]$ set of agents. Denote by L the set of all <u>strict</u> preferences over X.

Obs) We insist on strict preferences now.

Obs) L^n is the set of all preference profiles.

→ How do we "aggregate" individual agents' preferences?

Def) 1) Social Choice Function: A function $f : L^n \to X$.

2) Preference Aggregation Rule (PAR): A function $f : L^n \to L$.

Two Normative Properties: A PAR satisfies the <u>Pareto Principle</u> if $\forall \succsim \in L^n$, if $x \succsim_i y \ \forall i \in A$ then $x \ f(\succsim) \ y$. Denote $N(\succsim, x, y) := \{i \in A \mid x \succsim_i y\}$. A PAR satisfies <u>Independence of Irrelevant Alternatives</u> (IIA) if $\forall \succsim, \succsim' \in L^n$, if $N(\succsim, x, y) = N(\succsim', x, y)$, then $f(\succsim)$ should rank x and y the same as $f(\succsim')$.

Theorem) Arrow's Theorem: If $|X| \geq 3$, then a PAR f satisfies the Pareto Principle & IIA iff $\exists i \in A$ s.t. $f(\succsim) = \succsim_i \ \forall \succsim \in L^n$ (dictator).

Proof: $\Rightarrow$) Let $f: L^n \to L$ satisfy Pareto and IIA.

Let $B(x,y)$ be the set of all coalitions T for which $\exists \succsim \in L^n$ s.t. $N(\succsim, x, y) = T$ and $x f(\succsim) y$. Observe that $B(x,y)$ is never empty. By the Pareto Principle, A (coalition of all) is always in $B(x,y)$. Consider the set of all coalitions that belong to at least one $B(x,y)$ $\forall x \neq y \in X$. Let T be a minimal set in this collection. So, if $T' \subsetneq T \Rightarrow T'$ does not belong to any $B(x,y)$. We claim that such T is a singleton ($|T|=1$).

Suppose, towards a contradiction, that $|T| \geq 2$. Then, we can write $T = T_1 \cup T_2$ s.t. $T_1, T_2 \neq \emptyset$ and $T_1 \cap T_2 = \emptyset$. Let $a \neq b \in X$ s.t. $T \in B(a,b)$. Let $c \neq a,b$ (exists as $|X| \geq 3$), and consider a particular profile $\succsim \in L^n$ which is as follows: First, since $T \in B(a,b)$, $\exists \succsim'$ s.t. $N(\succsim', a, b) = T$

$\succsim_i \mid i \in T_1$	$\succsim_i \mid i \in T_2$	$\succsim_i \mid i \in A \setminus T$
a	c	b
b	a	c
c	b	a

and $a f(\succsim') b$. By IIA, $a f(\succsim) b$. Second, if we had $a f(\succsim) c \Rightarrow T_1 \in B(a,c)$, but this is impossible since T is a minimal set. So we must have $c f(\succsim) a$. By transitivity, $c \, f(\succsim) \, b$. So $T_2 \in B(c,b)$. $T_2 \subsetneq T$. Absurd.//

WLOG, let $T := \{1\}$. So $\{1\} \in B(a,b)$. Next, we claim that $\underline{\{1\} \in B(x,y) \; \forall x \neq y \in X}$. First, let $c \neq a,b$ be arbitrary. Consider the profile where: First, since $\{1\} \in B(a,b)$, $a \, f(\succsim) \, b$. Second, by

1	$A \setminus \{1\}$
c	b
a	c
b	a

Pareto, $c \, f(\succsim) \, a$. By transitivity, $c \, f(\succsim) \, b$. So $\{1\} \in B(c,b)$. Now consider some $d \neq b,c$ and the profile:

1	$A \setminus \{1\}$
c	b
b	d
d	c

Then, since $\{1\} \in B(c,b)$, by IIA, $c \, f(\succsim) \, b$. All agents rank b over d. So, by Pareto, $b \, f(\succsim) \, d$. So by transitivity, $c \, f(\succsim) \, d$.

But then, $\{1\} \in B(c,d)$. So $\{1\}$ is in every arbitrary $B(x,y)$.//

To finish, we shall prove that $\underline{\forall \succsim \in L^n, \text{ if } x \succsim_1 y, \text{ then } x \, f(\succsim) \, y}$.

Choose an arbitrary $x \neq y \in X$ and $\succsim \in L^n$. Suppose that $x \succsim_1 y$.

Consider a preference profile $\succsim' \in L^n$ and an outcome $z \neq x,y$ s.t. $x \succ'_1 z \succ'_1 y$ and $z \succ'_i x \succ'_i y \; \forall i \neq 1$ with $x \succ_i y$, and $z \succ'_i y \succ'_i x \; \forall i \neq 1$ with $y \succ_i x$. Since $\{1\} \in B(x,z)$, we have $x \, f(\succsim') \, z$.

$z \succ' y$ is unanimous, so by Pareto Principle, $z \, f(\succsim') \, y$. By transitivity, $x \, f(\succsim') \, y$. All agents rank x & y the same in $\succsim$ and

$\succsim'$, so by IIA, $x\, f(\succsim)\, y$. //

In the case of two alternatives: a social choice function, that chooses one outcome, is the same thing as a PAR.

Def) Permutation: a function $\sigma : A \to A$ that is one-to-one.

Def) Symmetric SCF: $\forall \succsim \in L^n$ and $\forall \sigma$ of A, $f(\succsim) = f((\succsim_{\sigma(i)})_{i \in A})$, i.e. the names of the agents don't matter, only presence of certain preferences.

Def) Monotonic SCF: $\forall x \neq y \in X = \{a, b\}$ and $\forall \succsim, \succsim' \in L^n$, if $f(\succsim) = x$ and any agent who ranks x over y in $\succsim$ also ranks x over y in $\succsim'$, then $f(\succsim') = x$.

Theorem) May's Theorem: Suppose $X = \{a, b\}$. A scf $f : L^n \to X$ (assume n is odd) is symmetric and monotonic iff $\exists q \in \mathbb{R}$ s.t. $f(\succsim) = \begin{cases} a & \text{if } |\{i \in A \mid a \succsim_i b\}| > q \\ b & \text{otherwise} \end{cases}$.

Proof: $\Rightarrow$) It is sufficient to prove that the decision to choose x over y is a monotone increasing function of the # of agents who prefer x over y. Let $\succsim \in L^n$ s.t. $f(\succsim) = x$ and let $\succsim' \in L^n$ be s.t. the # of agents who rank x over y is $\geq$ the # of agents who prefer x over y in $\succsim$. Now choose permutation σ s.t. anyone who ranks x over y in $\succsim$ also ranks x over y in $(\succsim'_{\sigma(i)})_{i \in A}$. We can do this because the # of such agents in $\succsim'$ is $\geq$ such agents in $\succsim$. By monotonicity, $x = f((\succsim'_{\sigma(i)})_{i \in A})$. By symmetry, $x = f(\succsim')$. //

$\Leftarrow$) In HW.

Incentives (Strategy Proofness)

Fix a set of outcomes X and a set of n agents. Let R be a set of all preferences over X (may not be strict). Fix a $\succsim \in R^n$. Then we obtain a particular social choice problem $(X, A, \{\succsim_i \mid i \in A\})$.

Def) Social Choice Function: a function $f: R^n \to X$. (now allowing R)

Def) Strategy Proofness: a scf s.t. $\forall \succsim \in R^n$, $\forall i \in A$ and $\forall \succsim'_i \in R$, $f(\succsim_i \cup \succsim \setminus \{\succsim_i\}) \succsim_i f(\succsim'_i \cup \succsim \setminus \{\succsim_i\})$ (or, $f(\succsim_i, \succsim_{-i}) \succsim_i f(\succsim'_i, \succsim_{-i})$), i.e. nobody is incentivized to lie about their real preferences.

Theorem) [Gibband & Satth.] Suppose that f is a scf s.t. its range ($f(R^n)$) has at least 3 elements. Then f is strategy-proof iff $\exists i \in A$ s.t. $\forall \succsim \in R^n$, $f(\succsim)$ is a top alternative for $\succsim_i$. (Proof omitted, similar to Arrow's)

↳ We can try to avoid dictatorship by <u>restricting the problem.</u>

ex) Let R_B be preferences that classify X into two equivalence classes,

the good / the bad, s.t. all elements in good are strictly preferred to those in bad. Now consider the scf $f: \mathcal{R}_B^n \to X$ for which $x = f(\succsim)$ means that the # of agents who regard x as good is maximized.
↳ This is strategy-proof because lying would only work against you!

Hardness of Manipulation: MANIP(f). Given $\succsim \in \mathcal{R}^n$, $x \in X$, and i, does there exist $\succsim_i'$ s.t. $x = f(\succsim_i', \succsim_{-i})$? → Can we find a scf that is computable by means of an efficient (polytime) algorithm w.r.t. n & $|X|$, s.t. MANIP(f) is (NP) hard?
ex) Single Transferrable Vote: Initialize $C \leftarrow X$. Let v_i be i's top choice in set C. If some alternative gets $\lfloor n/2 \rfloor + 1$ votes, choose it. Else, choose some alternative with the least votes, say x, and set $C \leftarrow C \setminus \{x\}$. Repeat until a winner is declared or all alternatives are exhausted.
↳ Let f^{STV} be a scf computed by STV. We claim that MANIP(f^{STV}) is hard. (proof omitted, use reduction from 3-SAT).

Incentives in Assignment Problems: Given an assignment problem $(A, O, \emptyset, \{\succsim_i | i \in A\})$, where $\succsim_i$ is a strict preference over $O^* := O \cup \{\emptyset\}$, let X be the set of all assignments, and say that $x \succsim_i x'$ if the element of O^* that i gets in x

is $\succsim_i$ to that in X'. In particular, a scf $f: L^n \to X$, where $L :=$ set of all strict preferences over O^*, is strategy-proof if $\forall \succsim \in L^n$, $\forall i \in A$, and $\forall \succsim_i' \in L$, $f(\succsim_i, \succsim_{-i}) \succsim_i f(\succsim_i', \succsim_{-i})$ when $f(\succsim)$ is what i gets in the assignment $f(\succsim)$.

Obs) f^{SD}, the scf obtained from serial dictatorship, is strategy-proof.
$\because$ lying about your preferences can only make you worse off.

Now consider a scf $f: L^n \to X$, and X is the set of all random assignments.
Def) Strategy-Proof: f s.t. $\forall \succsim \in L^n$, $i \in A$, $\succsim_i' \in L$, $f_i(\succsim)$ FOSD$_{\succsim_i}$ $f_i(\succsim_i', \succsim_{-i})$, i.e. the lottery that i gets by reporting $\succsim_i$ FOSD $\succsim_i'$ w.r.t. any $\underset{\text{value repr.}}{V(\succsim_i)}$.
Def) Weakly Strategy-Proof: f s.t. $\forall \succsim \in L^n$, $i \in A$, $\succsim_i' \in L$, $f_i(\succsim_i', \succsim_{-i})$ $\cancel{\text{FOSD}}_{\succsim_i}$ $f_i(\succsim_i, \succsim_{-i})$.
↳ issue: $f_i(\succsim_i', \succsim_{-i})$ and $f_i(\succsim_i, \succsim_{-i})$ may be incomparable (→ incomplete order) according to FOSD$_{\succsim_i}$, i.e. some value repr.s disagree. SP is much stronger than weakly SP!

Proposition) RSD is SP. PS is not SP but is weakly SP.
Proof of parts of prop: consider such instance.

$|O| = 3$, $n = 3$.

$\succsim_1$	$\succsim_2$	$\succsim_3$
o_1	o_2	o_2
o_2	o_1	o_3
o_3	o_3	o_1
$\emptyset$	$\emptyset$	$\emptyset$

The "Pacman" will eat as such:

	1	2	3
$1/2$	O_1	O_2	O_2
$1/4$	O_1	O_1	O_3
$1/4$	O_3	O_3	O_3

And the matrix will be:

$$X^{PS} = \begin{bmatrix} 3/4 & \emptyset & 1/4 \\ 1/4 & 1/2 & 1/4 \\ \emptyset & 1/2 & 1/2 \end{bmatrix}$$

Now, what happens if agent 1 misreports their preference as: $\succsim_1' = (O_2 > O_1 > O_3 > \emptyset)$?

	1	2	3
$1/3$	O_2	O_2	O_2
$1/2$	O_1	O_1	O_3
$1/6$	O_3	O_3	O_3

$$\Rightarrow \quad Z^{PS} = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ \emptyset & 1/3 & 2/3 \end{bmatrix}$$

If the upper contour set $U_{\succsim_1}(O_2) = \{O_1, O_2\}$, observe that under X^{PS}, this has probability $3/4$, while under Z^{PS}, it has probability $5/6$.

$\Rightarrow$ it is not the case that $f_i^{PS}(\succsim_1, \succsim_2, \succsim_3)$ FOSD$_i$ $f_i^{PS}(\succsim_1', \succsim_2, \succsim_3)$. Thus, PS is not SP.//

What about RSD? Let's write $SD_i(\succsim, \geq)$ for the object (or $\emptyset$) that i gets in SDA with profile $\succsim \in L^n$ and ordering $\geq$ of n agents.

Fix a utility $v_i : O^* \to \mathbb{R}$ for agent i. Then, their expected utility under

RSD with profile $\succsim \in L^n$ is $\sum_{\geq \in \Pi} \frac{1}{n!} v_i(SD_i(\succsim, \geq))$. (akin to SV)

Since SD is strategy-proof, no matter which $\succsim_i' \in L$ and $\geq \in \Pi$, $v_i(SD_i(\succsim, \geq)) \geq v_i(SD_i(\succsim', \geq))$. But since this is true for any ordering, we obtain: $\sum_{\geq \in \Pi} v_i(SD(\succsim, \geq)) \geq \sum_{\geq \in \Pi} v_i(SD_i(\succsim_i', \succsim_{-i}, \geq))$.

Since v_i was arbitrary, $RSD(\succsim)$ $FOSD_{\succsim_i}$ $RSD(\succsim_i', \succsim_{-i})$. //

Quasilinear Environments

(a.k.a. economic environments, "with money", or "with transfer")

Now, assume outcomes have a special structure: $x = (y, t_1, \dots, t_n)$ in which $y \in Y$, a set of decisions, and $t_i \in \mathbb{R}$ is a monetary transfer to agent i.

Ex1) Public Good: $Y = \{0, 1\}$. If $Y = 0$, then we don't build/acquire the public good. If $Y = 1$, then we do. Transfers t_i (positive or negative) may be required to cover the cost of building the public good.

* Sometimes we shall work with $Y = [0,1]$ and $y \in Y$ is a prob. of building.

Ex2) Private Good: A single indivisible good for sale. $Y = \{y \in \{0,1\}^n \mid \sum_{i=1}^{n} y_i = 1\}$ (one-hot for who gets the good).

* Similarly, we sometimes work with $Y = \{y = [0,1]^n \mid \sum_{i=1}^{n} y_i = 1\}$, s.t. y is a probability distribution.

Formally, a social choice problem $(X, A, \{\succsim_i \mid i \in A\})$ is a <u>quasilinear environment</u> if: 1) $X = (Y \times \mathbb{R}^n)$ with Y being the set of possible decisions

2) Each $\succsim_i$ can be represented by a utility function of the form:

$$u_i(x) = u_i(y, t_1, \dots, t_n) = v_i(y_i) + t_i \text{ where } v_i : Y \to \mathbb{R}.$$

Def) Pareto Optimality: an outcome $(y, t_1, \dots, t_n)$ is Pareto Dominated if $\exists$ another outcome $(y', t_1', \dots, t_n')$ s.t. $v_i(y') + t_i' \geq v_i(y) + t_i \ \forall i \in A$ while $\sum_{i=1}^{n} t_i' \leq \sum_{i=1}^{n} t_i$ (money is constrained). x is PO if $\nexists$ an outcome that PDs it.

Def) Efficient (Welfare Maximising) Decision: $y \in Y$ s.t. $\sum_{i=1}^{n} v_i(y) \geq \sum_{i=1}^{n} v_i(y')$ $\forall y' \in Y$, i.e. y solves the problem of $\max_{y \in Y} \{\sum_{i=1}^{n} v_i(y)\}$.

Claim) $(y, t_1, \dots, t_n)$ is PO iff y is efficient.

Proof: $\Leftarrow$) By contrapositive, suppose $(y, t_1, \dots, t_n)$ is not PO. We shall prove that Y is not efficient. Since not PO, $\exists (y', t_1', \dots, t_n')$ with $v_i(y) + t_i < v_i(y') + t_i' \ \forall i \in A$ while $\sum_{i=1}^{n} t_i \geq \sum_{i=1}^{n} t_i'$. So we have $v_i(y') > v_i(y) + [t_i - t_i'] \ \forall i$, and summing over i, $\underline{\sum_{i=1}^{n} v_i(y')} > \sum_{i=1}^{n} v_i(y) + \sum_{i=1}^{n} [t_i - t_i'] \geq \underline{\sum_{i=1}^{n} v_i(y)}$, which means y is not efficient. //

$\Rightarrow$) Let $(y, t_1, \dots, t_n)$ be an outcome in which y is not efficient. We shall prove that $(y, t_1, \dots, t_n)$ is not PO.

Then, by definition, suppose that $y' \in Y$ s.t. $\sum_{i=1}^n v_i(y) < \sum_{i=1}^n v_i(y')$. Let $\varepsilon > 0$ be s.t. $\sum_{i=1}^n v_i(y) + \varepsilon < \sum_{i=1}^n v_i(y')$. Define transfers $t_i' := v_i(y) + t_i - v_i(y') + \frac{\varepsilon}{n}$. Now, $v_i(y') + t_i' = v_i(y) + t_i + \frac{\varepsilon}{n} > v_i(y) + t_i \ \forall i$, while $\sum_{i=1}^n t_i' = \underbrace{\sum_{i=1}^n v_i(y) - \sum_{i=1}^n v_i(y') + \varepsilon}_{} \overset{<0}{} + \sum_{i=1}^n t_i < \sum_{i=1}^n t_i$. Hence, $(y, t_1, \dots, t_n)$ is not PO.

Let V be a set of functions $v: Y \to \mathbb{R}$, then we may define a scf $f: V^n \to X$. The interpretation is that when agents' utilities over decisions are $(v_1, \dots, v_n)$ with $v \in V$, then the outcome depends on them, i.e., $f(v_1, \dots, v_n) = (y(v_1, \dots, v_n), t_1(v_1, \dots, v_n), \dots, t_n(v_1, \dots, v_n))$. The function $y(\cdot): V^n \to Y$ is a <u>decision rule</u>, and the functions $t_i(\cdot): V^n \to \mathbb{R}$ is a <u>transfer rule</u> $\forall i \in [n]$.

Ex) Private Good Environment with $Y = \{(y_1, \dots, y_n) \in [0,1]^n \mid \sum_{i=1}^n y_i = 1\}$.

Assume a class of utilities V s.t. $\forall v \in V$, $\exists \vec{\theta} \in [0,1]^n$ s.t. $v_i(\vec{y}) = \theta_i y_i$.

The number θ_i is the <u>valuation</u> that agent i has has for the good.

↳ One scf is "posted prize." Fix a price $p \in [0,1]$. $y_i(\theta_1, \dots, \theta_n) = \begin{cases} 1/J & \text{if } \theta_i \geq p \\ 0 & \text{if } \theta_i < p \end{cases}$ where $J := |\{j \mid \theta_j \geq p\}|$, $t_i(\theta_1, \dots, \theta_n) = -y_i(\theta_1, \dots, \theta_n) p$.

↳ Another scf is the "first-price sealed bid" auction. $y_i(\vec{\theta}) = \mathbb{1}\{\theta_i = \max_i \{\theta_i\}\}$.

$t_i(\vec{\theta}) = y_i(\vec{\theta}) \cdot \theta_i$.

↳ A third example is a "second-price sealed bid" auction. $y_i(\vec{\theta}) = 1\{\theta_i = \max_i\{\theta_i\}\}$.

However, the transfer rule is $t_i(\vec{\theta}) = y_i(\vec{\theta}) \cdot \max_j\{\theta_j \mid j \neq i\}$.

Def) Efficiency for decision rule: $y(\cdot)$ s.t. $\forall (v_1, \dots, v_n)$, $\sum_{i=1}^{n} v_i(y) \geq \sum_{i=1}^{n} v_i(\tilde{y})\ \forall \tilde{y}$.

↳ i.e. one that always make efficient decisions.

Def) Vickney-Clarke-Groves (VCG) Transfer Rule: $t_i(\cdot)$ s.t. $t_i(v_1, \dots, v_n)$ $= \sum_{j \neq i} v_j(y^*(v_1, \dots, v_n)) + h_i(v_{-i})$ where y^* is an efficient decision rule and $h_i : \times_{j \neq i} V_j \to \mathbb{R}$ is a function that only depends on v_j for $j \neq i$.

Ex) $Y = \{y_1, y_2, y_3\}$, $n = 2$. Valuation are:

	y_1	y_2	y_3
v_1	20	18	12
v_2	14	18	21

$\to y_2$ maximizes $v_1 + v_2$.

↳ VCG transfer is $t_1(v_1, v_2) = 18 + h_1(v_2)$, $t_2(v_1, v_2) = 18 + h_2(v_1)$

Def) VCG Mechanism: scf $f : V^n \to Y \times \mathbb{R}^n$ s.t. its decision rule is efficient and each agent's transfer rule is VCG.

Claim) Any VCG mechanism is strategy-proof.

Proof: Fix a profile $(v_1, \dots, v_n) \in V^n$. Consider agent i and a possible misrepresentation $v_i' \in V$. By reporting v_i truthfully, i obtains utility

$$v_i(y^*(v_1, \dots, v_n)) + t_i(v_1, \dots, v_n) = v_i(y^*(v_1, \dots, v_n)) + \sum_{j \neq i} v_j(y^*(v_1, \dots, v_n)) + h_i(v_{-i})$$

$$= \underbrace{\sum_{j=1}^{n} v_j(y^*(v_1, \dots, v_n)) + h_i(v_{-i}) \geq \sum_{j=1}^{n} v_j(y^*(v_i', v_{-i}))}_{y^* \text{ is efficient}} + h_i(v_{-i}) = v_i(y^*(v_i', v_{-i})) + t_i(v_i', v_{-i}).$$ //

Def) Pivot Rule: a VCG transfer rule of the form $t_i(v_i,\dots,v_n)=\underbrace{\sum_{j\neq i} v_j(y^*)}_{\text{welfare of other agents at an eff. decision with } i}$ $-\overbrace{\max\{\sum_{j\neq i} v_j(\tilde{y}) \mid \tilde{y}\in Y\}}^{\text{welfare of other agents at an eff. decision without } i}$.

Ex)

	y_1	y_2	y_3
v_1	20	18	12
v_2	14	18	21

$t_1(v_1,v_2)=18-21=-3$, $t_2(v_1,v_2)=18-20=-2$.

Application) Single indivisible private good: $Y=\{(y_1,\dots,y_n)\in[0,1]^n \mid \sum_{i=1}^n y_i=1\}$ and each v_i is associated with a scalar $\theta_i\geq 0$ s.t. $v_i(y_1,\dots,y_n)=\theta_i y_i$.

↳ Efficient y solves $\max_y \sum_{i=1}^n \theta_i y_i$ s.t. $y_i\in[0,1]$, $\sum_{i=1}^n y_i=1$. So $y_i^*(\theta_1,\dots,\theta_n)>0$ only if $\theta_i=\max\{\theta_j \mid 1\leq j\leq n\}$. → Pivot rule gives the <u>second price auction</u>!

$t_i(\theta_1,\dots,\theta_n)=\sum_{j=1}^n \theta_j y_j^*(\theta_1,\dots,\theta_n)-\max\{\sum_{j\neq i}\theta_j\tilde{y}_j \mid 0\leq\tilde{y}_j\leq 1\ \forall j\neq i, \sum_{j\neq i}\tilde{y}_j=1\}$. So if $\theta_i>\theta_j\ \forall j\neq i$, i's utility is $\theta_i\overbrace{y_i^*(\theta_1,\dots,\theta_n)}^{1}+\overbrace{\sum_{j\neq 1}\theta_j y_j^*(\theta_1,\dots,\theta_n)}^{0}-\overbrace{\max\{\theta_j \mid j\neq i\}}$ $=\theta_i-\max\{\theta_j \mid \theta_j<\theta_i\}$. Otherwise, if $\theta_i<\theta_{j_0}$ for some j_0, then i's utility is $\theta_i\overbrace{y^*(\theta_1,\dots,\theta_n)}^{0}+\theta_{j_0}-\theta_{j_0}=0$. //

Application) Google's ad pricing: Model with k slots. Each slot has a click-through rate (CTR) $\alpha_k>0$. Ordering is $\alpha_1>\alpha_2>\dots>\alpha_k$. There are n bidders, $n>k$. Each has a valuation $\theta_i>0$ for having their ad clicked on. Assume, for notations, that we have extra artificial slots with 0 CTR, $\alpha_{k+1}=\alpha_{k+2}=\dots=\alpha_n=0$. A decision y is an assignment of slots to bidders. We try to maximize $\sum_{i=1}^{n} \alpha_k \theta_{y_k}$ where y_k is the bidder who gets slot k. Observe that if $\alpha_k > \alpha_{k'}$ and $\theta_i > \theta_j$, then $\alpha_k \theta_j + \alpha_{k'} \theta_i < \alpha_k \theta_i + \alpha_k \theta_j$. So, an efficient decision will label the bidders such that $\theta_1 > \theta_2 > \dots > \theta_n$, and slot k goes to bidder k.

Then, the VCG pivot payments will be (for bidder i):

$$t_i(\theta_1, \dots, \theta_n) = \overbrace{\sum_{j<i} \alpha_j \theta_j + \sum_{j>i} \alpha_j \theta_j}^{\sum_{j \neq i} v_j(y^*)} - \overbrace{\left[\sum_{j<i} \alpha_j \theta_j + \sum_{j>i} \alpha_{j-1} \theta_j \right]}^{\max\{\sum_{j \neq i} v_j(\tilde{y}) \mid \tilde{y} \in Y\}} = \underbrace{\sum_{j>i} (\alpha_j - \alpha_{j-1}) \theta_j}_{<0}.$$

↳ this is called a "generalized second-price auction" (GSA)

Obs) GSA is not strategy-proof. Consider $k=2$, $n=3$, $\alpha_1 = 0.2$, $\alpha_2 = 0.199$, $\theta_1 = 10$, $\theta_2 = 6$, $\theta_3 = 2$. If 1 bids truthfully, they get $0.2(10-6) = 0.8$ utility. But if 1 reports some value $6 > b_1 > 2$, they get $0.199(10-2) \simeq 1.6$, which is clearly the better option. //

Combinatorial Auctions

The model: set of items G, n bidders, each bidder i has a valuation function $v: 2^{|G|} \to \mathbb{R}$ and quasilinear preferences $v_i(A) + t_i$ if they are awarded $A \subseteq G$ and transfer is t_i. An assignment is a collection $A_1, A_2, \dots, A_n$ disjoint subsets. Note that some A_i might be empty.

ex) railroad segments, radio spectrum auctions

Let Y be the set of all assignments. Then, we may write the utility of agent i at outcome $(y, t_1, \dots, t_n)$ as $V_i(y) + t_i = V_i(A_i) + t_i$. The wellfare maximization is $\max \sum_{i=1}^{n} V_i(A_i)$ s.t. $\{A_i \subseteq G,\ A_i \cap A_j = \emptyset\}$.

Def) Single Minded: bidder with valuation V_i where $\exists B_i \subseteq G$ and $\gamma_i > 0$ s.t. $V_i(A) = \gamma_i \cdot \mathbb{1}\{B_i \subseteq A\}$.

Claim) Suppose all bidders are single minded. Then, $\overbrace{\text{welfare maximization}}^{\max \sum_{i=1}^{n} V_i(A_i)}$ problem is NP-Hard.

Proof: Reduction from k-Independent Set $\to$ Well. Max.

First, we consider the decision problem. Given $v_1, \dots, v_n$ and $k \in \mathbb{R}$, is there an assignment $(A_1, \dots, A_n)$ s.t. $\sum_{i=1}^{n} v_i(A_i) \geq k$?

Given a graph $G(V,E)$, an independent set is a subset $I \subseteq V$ s.t. they have no edges between them. $\to$ is there ind. set of size $\geq k$?

↳ This problem is famously NP-complete.

Fix an instance $G(V,E), k$ of ind. set. Define an instance of Well. Max with single minded bidders by: Let V be the set of bidders $\{1, \dots, n\}$ and $G \leftarrow E$. For each bidder i, let $\gamma_i = 1$ and B_i be the set of edges that are incident to it. Observe that if $(A_1, \dots, A_n)$ is an assignment, then $\sum_{i=1}^{n} v_i(A_i) = \#$ of agents who obtain $A_i \supseteq B_i$. These agents form an independent set because if $(i,j) \in E$ and $A_i \supseteq B_i$, then $(i,j) \in B_j$ while $(i,j) \notin A_j$, so $v_j(A_j) = 0$. Conversely, from each ind. set of size $\geq k$, we may obtain an allocation with $\sum_{i=1}^{n} v_i(A_i) \geq k$. Thus, $\exists (A_1, \dots, A_n)$ s.t. $\sum_{i=1}^{n} v_i(A_i) \geq k$ iff $\exists$ an ind. set with cardinality $\geq k$. Reduction is complete. //

Revenue

Motivation: Selling a single indivisible item. Buyers have valuations $\theta \geq 0$, their willingness to pay for the item. $(F(\theta) = \int_{-\infty}^{\theta} f(s)ds)$

↳ Assume that $\theta \sim F$ where F is some CDF on $\mathbb{R}^+$.

For a fixed price p, a buyer will accept the price iff $\theta \geq p$.

↳ Expected revenue is $\underline{p(1-F(p))}$. (assume $f > 0$ on supp(F))

* We assume F is absolutely continuous with a density f where $F' = f$.

Then, the maximum revenue is found by the first order condition,

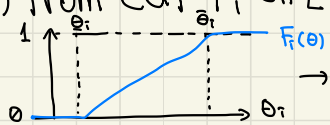
$$\frac{\partial}{\partial p}\left[p(1-F(p))\right] = 1 - F(p) - pf(p) = 0 \Rightarrow \underline{p^* = \frac{1-F(p^*)}{f(p^*)}}.$$

Remarks: $(1-F(p))$ can be thought of Demand(p), and $1-F(p)-pf(p)$ is a Marginal Revenue. Marginal Cost $=0$, so $\frac{\partial}{\partial p}[p(1-F(p))]=0$ works.

Def) Regular: F s.t. $\frac{1-F(p)}{f(p)}-p$ is monotone decreasing.

↳ If F is regular, then there is a unique optimal price p^*.

General Setting: Single indivisible good, n buyers. QLE of $(Y,\mathbb{R}^n)$ with $Y=\{(y_1,\dots,y_n)\in[0,1]^n \mid \sum_{i=1}^n y_i\le 1\}$ w.p. $1-\sum_{i=1}^n y_i$ of seller keeping the good. Assume each buyer i has valuation $\theta_i\in[\underline{\theta}_i,\bar{\theta}_i]\subseteq\mathbb{R}^+$ s.t. their utility from outcome $(y,t_1,\dots,t_n)$ is $\theta_i y_i+t_i$. Also assume that each θ_i is drawn (independently) from cdf F_i on $[\underline{\theta}_i,\bar{\theta}_i]$ with density f_i where $f_i>0$ on $[\underline{\theta}_i,\bar{\theta}_i]$.



→ $F_i(\hat{\theta}_i)$: Pr[randomly drawn buyer has $\theta_i\le\hat{\theta}_i$]

A scf is a function $g(\theta_1,\dots,\theta_n)=(y_1(\vec{\theta}),\dots,y_n(\vec{\theta}),t_1(\vec{\theta}),\dots,t_n(\vec{\theta}))$.

The seller's revenue is $\sum_{i=1}^n(-t_i(\vec{\theta}))$, but the seller doesn't know θ_is.

→ Expected revenue is $\int_{\underline{\theta}_1}^{\bar{\theta}_1}\int_{\underline{\theta}_2}^{\bar{\theta}_2}\cdots\int_{\underline{\theta}_n}^{\bar{\theta}_n}\sum_{i=1}^n[-t_i(\tilde{\theta}_1,\dots\tilde{\theta}_n)]dF_n(\tilde{\theta}_n)\cdots\overset{=f_1(\tilde{\theta}_1)d\tilde{\theta}_1}{dF_1(\tilde{\theta}_1)}$.

Similarly, expected utility of agent i is $\int_{\underline{\theta}_i}^{\bar{\theta}_i}\int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}}[\tilde{\theta}_i y_i(\tilde{\theta}_i,\tilde{\theta}_{-i})+t_i(\tilde{\theta}_i,\tilde{\theta}_{-i})]f_i(\tilde{\theta}_i)f_{-i}(\tilde{\theta}_{-i})\,d\tilde{\theta}_{-i}\,d\tilde{\theta}_i$. This is referred to as an <u>ex-ante utility</u>, when all agents report their valuations truthfully.

The <u>interim</u> expected payoff (to agent i) after knowing that their valuation is θ_i is: $U_i(\theta_i,\theta_i) = \int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}} [\theta_i y(\theta_i,\tilde{\theta}_{-i}) + t_i(\theta_i,\tilde{\theta}_{i})] f_{-i}(\hat{\theta}_{-i}) d\hat{\theta}_{-i}$.

However, the IEP to i after knowing that their valuation is θ_i but reporting their valuation as θ_i' is $U_i(\theta_i,\theta_i') = \int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}} [\theta_i y(\theta_i',\tilde{\theta}_{-i}) + t_i(\theta_i',\tilde{\theta}_{i})] f_{-i}(\hat{\theta}_{-i}) d\hat{\theta}_{-i}$.

Now, since expectation is linear, this is $\theta_i \underbrace{\int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}} y(\theta_i',\hat{\theta}_{-i}) f_{-i}(\hat{\theta}_{-i}) d\tilde{\theta}_{-i}}_{=:V_i(\theta_i')} + \underbrace{\int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}} t_i(\theta_i',\tilde{\theta}_{-i}) f_{-i}(\tilde{\theta}_{-i}) d\hat{\theta}_{-i}}_{=:T_i(\theta_i')} = \theta_i V_i(\theta_i') + T_i(\theta_i')$.

Def) Incentive Compatible: A scf $(y(\vec{\theta}), t_1(\vec{\theta}), \dots, t_n(\vec{\theta}))$ that $\forall i \in [n]$, $\forall \theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$, $\forall \theta_i' \in [\underline{\theta}_i, \bar{\theta}_i]$, $U_i(\theta_i,\theta_i) \geq U_i(\theta_i,\theta_i')$.

Obs) $U_i(\theta_i,\theta_i) = \theta_i V_i(\theta_i) + t_i(\theta_i) \geq \theta_i V_i(\theta_i') + t_i(\theta_i') = U_i(\theta_i,\theta_i')$.

$U_i(\theta_i',\theta_i') = \theta_i' V_i(\theta_i') + t_i(\theta_i') \geq \theta_i' V_i(\theta_i) + t_i(\theta_i) = U_i(\theta_i',\theta_i)$ must be true $\forall \theta_i, \theta_i' \in [\underline{\theta}_i, \bar{\theta}_i]$ if the scf is incentive compatible. Adding the inequalities,

$\theta_i V_i(\theta_i) + \cancel{t_i(\theta_i)} + \theta_i' V_i(\theta_i') + \cancel{t_i(\theta_i')} \geq \theta_i V_i(\theta_i') + \cancel{t_i(\theta_i')} + \theta_i' V_i(\theta_i) + \cancel{t_i(\theta_i)}$

$\rightarrow \theta_i(V_i(\theta_i) - V_i(\theta_i')) + \theta_i'(V_i(\theta_i') - V_i(\theta)) \geq 0 \Rightarrow (\theta_i - \theta_i')(V_i(\theta_i) - V_i(\theta_i')) \geq 0$.

Claim) If scf is incentive compatible, then $\forall i$, $V_i(\cdot)$ is monotone nondecreasing (weakly increasing), i.e. $\theta_i \leq \theta_i' \Rightarrow V_i(\theta_i) \leq V_i(\theta_i')$. (Proof by last line above)

Lemma) If scf is IC, then $U_i^*(\theta_i) := U_i(\theta_i,\theta_i)$ is convex and almost everywhere differentiable. If it is differentiable at $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$, then its derivative is $\frac{\partial}{\partial\theta_i} U_i^*(\theta_i) = V_i(\theta_i)$.

Proof: (Sketch) If scf is IC, then $U_i^*(\theta_i) \geq U_i(\theta_i, \theta_i') \ \forall \theta_i' \in [\underline{\theta}_i, \bar{\theta}_i]$.

So, $U_i^*(\theta_i) = \max_{\theta_i'} \{U_i(\theta_i, \theta_i')\} = \max_{\theta_i'} \{\theta_i V_i(\theta_i') + T_i(\theta_i')\} \Rightarrow U_i^*(\theta_i)$ is convex.

It follows that $U_i^*(\theta_i)$ is almost everywhere differentiable. //

Suppose that $\theta_i \in (\underline{\theta}_i, \bar{\theta}_i)$ is a point of differentiability. Then, $\frac{\partial}{\partial\theta_i} U_i^*(\theta_i) = \lim_{h\to 0} \frac{U_i^*(\theta_i+h) - U_i^*(\theta_i)}{h}$. Now, $U_i^*(\theta_i+h) - U_i^*(\theta_i) \overset{(\text{by IC})}{\geq} U_i(\theta_i+h, \theta_i) - U_i(\theta_i, \theta_i)$

$= (\theta_i+h)V_i(\theta_i) + T_i(\theta_i) - [\theta_i V_i(\theta_i) + T_i(\theta_i)] = hV_i(\theta_i)$. If $h>0$, then $\frac{U_i^*(\theta_i+h) - U_i^*(\theta_i)}{h} \geq V_i(\theta_i)$, and if $h<0$, $\leq V_i(\theta_i)$. Thus, $V_i(\theta_i) \leq \lim_{h\to 0^+} \frac{U_i^*(\theta_i+h) - U_i^*(\theta_i)}{h} = \frac{\partial}{\partial\theta_i} U_i^*(\theta_i) = \lim_{h\to 0^-} \frac{U_i^*(\theta_i+h) - U_i^*(\theta_i)}{h} \leq V_i(\theta_i)$. By the squeeze theorem, $\frac{\partial}{\partial\theta_i} U_i^*(\theta_i) = V_i(\theta_i)$. //

Corollary: If the scf is IC, then $U_i^*(\theta_i) = U_i^*(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds$.

Obs) We have $\theta_i V_i(\theta_i) + T_i(\theta_i) = U_i^*(\theta_i) = U_i^*(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds$. Then, $T_i(\theta_i) = U_i^*(\underline{\theta}_i) - \theta_i V_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds$. So the transfers are pinned down by $U_i^*(\underline{\theta}_i)$ and $V_i(\cdot)$.

Lemma) A scf is IC iff: 1) $V_i(\cdot)$ is monotone weakly increasing, and 2) $T_i(\theta_i) = U_i^*(\underline{\theta}_i) - \theta_i V_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds \ \ \forall i, \forall \theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$.

Proof: Necessity follows from the previous lemma & corollary. We prove sufficiency here. Fix a scf and assume conditions 1) and 2). Fix i, θ_i, θ_i'. Then $U_i(\theta_i,\theta_i) - U_i(\theta_i,\theta_i') = \theta_i V_i(\theta_i) + T_i(\theta_i) - [\theta_i V_i(\theta_i') + T_i(\theta_i')] = \theta_i[\cancel{V_i(\theta_i)} - V_i(\theta_i')] + [\cancel{U_i^*(\underline{\theta}_i)} - \cancel{\theta_i V_i(\theta_i)} + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds] - [\cancel{U_i^*(\underline{\theta}_i)}$ $- \theta_i' V_i(\theta_i') + \int_{\underline{\theta}_i}^{\theta_i'} V_i(s)\,ds] = V_i(\theta_i')(\theta_i' - \theta_i) + \int_{\theta_i'}^{\theta_i} V_i(s)\,ds \geq V_i(\theta_i')(\theta_i' - \theta_i)$

↳ due to monotonicity

$+ \int_{\theta_i'}^{\theta_i} V_i(\theta_i')\,ds = V_i(\theta_i')(\theta_i' - \theta_i) + V_i(\theta_i')(\theta_i - \theta_i') = 0. \Rightarrow \underline{U_i(\theta_i,\theta_i) \geq U_i(\theta_i,\theta_i')}$ //

Our problem: $\max_{f:\text{scf}} \left\{ \int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\bar{\theta}_n} \sum_{i=1}^{n} (-t_i(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)) \prod_{i=1}^{n} f_i(\tilde{\theta}_i)\, d\tilde{\theta}_n \cdots d\tilde{\theta}_1 \right\}$ s.t. $f = (y(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is IC and $\underbrace{U_i^*(\theta_i) \geq 0 \ \forall \theta_i \in [\underline{\theta}_i, \bar{\theta}_i] \ \forall i \in [n]}$. incentive to participate

$\rightarrow \int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\bar{\theta}_n} \sum_{i=1}^{n} (-t(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)) \prod_{i=1}^{n} f_i(\tilde{\theta}_i)\, d\tilde{\theta}_n \cdots d\tilde{\theta}_1 = \sum_{i=1}^{n} \int_{\underline{\theta}_1}^{\bar{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\bar{\theta}_n} (-t(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)) \prod_{i=1}^{n} f_i(\tilde{\theta}_i)\, d\tilde{\theta}_n \cdots d\tilde{\theta}_1$

$= \sum_{i=1}^{n} \int_{\underline{\theta}_i}^{\bar{\theta}_i} \int_{\underline{\theta}_{-i}}^{\bar{\theta}_{-i}} (-t(\tilde{\theta}_1,\ldots,\tilde{\theta}_n)) f_i(\tilde{\theta}_i) f_i(\tilde{\theta}_{-i})\, d\tilde{\theta}_{-i}\, d\tilde{\theta} = \sum_{i=1}^{n} \int_{\underline{\theta}_i}^{\bar{\theta}_i} (-T(\tilde{\theta}_i)) f_i(\tilde{\theta}_i)\, d\tilde{\theta}_i.$

$\rightarrow$ So our problem becomes $\max \left\{ \sum_{i=1}^{n} \int_{\underline{\theta}_i}^{\bar{\theta}_i} [-T_i(\tilde{\theta}_i) f_i(\tilde{\theta}_i)\, d\tilde{\theta}_i] \right\}$ subject to:

1) $V_i(\cdot)$ is monotone increasing, 2) $T_i(\tilde{\theta}_i) = U_i^*(\underline{\theta}_i) - \theta_i V_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} V(s)\,ds$,

3) $U_i^*(\theta_i) = U_i^*(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds \geq 0$.

$\rightarrow$ Observe that participation constraints are equivalent to $U_i^*(\underline{\theta}_i) \geq 0$ b/c participation has to be satisfied for $\theta_i = \underline{\theta}_i$, and once $U_i^*(\underline{\theta}_i) \geq 0$, then $U_i^*(\theta_i) \geq 0 \ \forall \theta_i \geq \underline{\theta}_i$ since $V_i(s) \geq 0 \ \forall s$.

$\rightarrow$ So our problem is: $\max \left\{ \sum_{i=1}^{n} \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[-U_i^*(\underline{\theta}_i) + \tilde{\theta}_i V_i(\tilde{\theta}_i) - \int_{\underline{\theta}_i}^{\tilde{\theta}_i} V_i(s)\,ds \right] f_i(\tilde{\theta}_i)\, d\tilde{\theta}_i \right.$

subject to: 1) $v_i(\cdot)$ is weakly monotone increasing, 2) $U_i(\underline{\theta}_i)\geq 0\ \forall i$.

→ Observe that an optimal scf will have $U_i(\underline{\theta}_i)=0$. Otherwise, we can find $\varepsilon>0$ and set $t_i(\cdot)-\varepsilon$ to be the transfer to i to get more revenue.

→ Let's calculate first: $\int_{\underline{\theta}_i}^{\bar{\theta}_i}\overbrace{f_i(\tilde{\theta}_i)}^{dv}\overbrace{\int_{\underline{\theta}_i}^{\tilde{\theta}_i}v_i(s)ds}^{u}\,d\tilde{\theta}_i$. Integration by parts:

$F_i(\hat{\theta}_i)\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(s)ds\Big|_{\underline{\theta}_i}^{\bar{\theta}_i}-\int_{\underline{\theta}_i}^{\bar{\theta}_i}F_i(\hat{\theta}_i)v_i(\tilde{\theta}_i)d\tilde{\theta}_i$. Since $F_i(\underline{\theta}_i)=0$, $F_i(\bar{\theta}_i)=1$, this becomes $\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(s)ds-\int_{\underline{\theta}_i}^{\bar{\theta}_i}F_i(\hat{\theta}_i)v_i(\tilde{\theta}_i)d\tilde{\theta}_i=\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(\hat{\theta}_i)(1-F_i(\hat{\theta}_i))\cdot\frac{f_i(\tilde{\theta}_i)}{f_i(\hat{\theta}_i)}d\hat{\theta}_i$.

Then, $\int_{\underline{\theta}_i}^{\bar{\theta}_i}\left[\tilde{\theta}_i v_i(\tilde{\theta}_i)-\int_{\underline{\theta}_i}^{\tilde{\theta}_i}v_i(s)ds\right]f_i(\tilde{\theta}_i)\,d\tilde{\theta}_i=\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(\hat{\theta}_i)\left[\theta_i-\frac{1-F_i(\hat{\theta}_i)}{f_i(\hat{\theta}_i)}\right]f_i(\hat{\theta}_i)d\hat{\theta}_i$.

→ Now, our problem is: $\max\left\{\sum_{i=1}^{n}\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(\tilde{\theta}_i)\left[\hat{\theta}_i-\frac{1-F_i(\hat{\theta}_i)}{f_i(\hat{\theta}_i)}\right]f_i(\hat{\theta}_i)d\hat{\theta}_i\right\}$ subject to: $v_i(\cdot)$ is weakly monotone increasing.

→ Let $\varphi_i(\theta_i):=\theta_i-\frac{1-F_i(\theta_i)}{f_i(\theta_i)}$, which is i's <u>virtual valuation</u>.

→ Transform the objective using Fubini: $\sum_{i=1}^{n}\int_{\underline{\theta}_i}^{\bar{\theta}_i}v_i(\hat{\theta}_i)\cdot\varphi_i(\tilde{\theta}_i)\cdot f_i(\hat{\theta}_i)d\tilde{\theta}_i$

$$=\sum_{i=1}^{n}\int_{\underline{\theta}_1}^{\bar{\theta}_1}\cdots\int_{\underline{\theta}_n}^{\bar{\theta}_n}y_i(\tilde{\theta}_1,\dots,\tilde{\theta}_n)\cdot\varphi_i(\hat{\theta}_i)\cdot\prod_{j=1}^{n}f_j(\tilde{\theta}_j)\,d\tilde{\theta}_n\cdots d\tilde{\theta}_1$$

$$=\int_{\underline{\theta}_1}^{\bar{\theta}_1}\cdots\int_{\underline{\theta}_n}^{\bar{\theta}_n}\sum_{i=1}^{n}y_i(\tilde{\theta}_1\dots\tilde{\theta}_n)\,\varphi_i(\tilde{\theta}_i)\cdot\prod_{j=1}^{n}f_j(\tilde{\theta}_j)\,d\hat{\theta}_n\cdots d\tilde{\theta}_1.$$

→ Consider a relaxed problem that drops the constraint that $v_i(\cdot)$ be monotone increasing. Then, we have:

$\max\left\{\int_{\underline{\theta}_1}^{\bar{\theta}_1}\cdots\int_{\underline{\theta}_n}^{\bar{\theta}_n}\sum_{i=1}^{n}y_i(\tilde{\theta}_1\dots\tilde{\theta}_n)\,\varphi_i(\tilde{\theta}_i)\cdot\prod_{j=1}^{n}f_j(\tilde{\theta}_j)\,d\tilde{\theta}_n\cdots d\tilde{\theta}_1\right\}$ subject to:

$y_i(\theta_1,\dots,\theta_n)\in[0,1]$ and $\sum_{i=1}^{n}y_i(\theta_1,\dots,\theta_n)\leq 1\quad\forall i,\ \forall(\theta_1,\dots,\theta_n)$.

→ Solve the relaxed problem pointwise ($\vec{\theta}$ by $\vec{\theta}$):

$\max\left\{\sum_{i=1}^{n} y_i(\theta_1,\dots,\theta_n)\,\varphi_i(\theta_i)\right\}$ subject to: $y_i(\cdot)\in[0,1]$, $\sum_{i=1}^{n} y_i \leq 1$

for each fixed vector of valuations $(\theta_1,\dots,\theta_n)$.

→ The solution to the relaxed problem is: $y_i(\theta_1,\dots,\theta_n) = \mathbb{1}\{\max_i \varphi_i(\theta_i) > 0\}$.

→ Assumption: The distributions F_i $\forall i$ are regular if $\varphi_i(\theta_i) = \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$ are all strictly monotone increasing.

→ Now, $V_i(\theta_i) = \int_{\underline{\theta}_{-i}}^{\overline{\theta}_{-i}} y_i(\theta_i,\tilde{\theta}_{-i})\,dF_i(\tilde{\theta}_{-i}) = \Pr[\varphi_i(\theta_i) > 0 \wedge \varphi_i(\theta_i) > \varphi_j(\tilde{\theta}_j)\ \forall j\neq i]$,

which is monotone increasing in θ_i when $\varphi_i(\cdot)$ is monotone increasing.

→ Thus, under regularity, any solution to the relaxed problem is also a solution to our original problem.

→ Let $Y_i(\theta_{-i}) := \min\{\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \mid \varphi_i(\theta_i) \geq 0 \wedge \varphi_i(\theta_i) > \varphi_j(\theta_j)\ \forall j\neq i\}$.

Then $y_i(\theta_i,\theta_{-i}) = 1 \Leftrightarrow \theta_i > Y_i(\theta_{-i})$.

→ Recall that interim transfers $T_i(\theta_i) = \cancel{U_i(\underline{\theta}_i)}^{\,0} - \theta_i V_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i} V_i(s)\,ds$.

Let $t_i(\theta_i,\theta_{-i}) = -\theta_i y_i(\theta_i,\theta_{-i}) + \underbrace{\int_{\underline{\theta}_i}^{\theta_i} y_i(s,\theta_{-i})\,ds}_{} \to \mathbb{1}\{\theta_i > Y_i(\theta_{-i})\}\int_{Y_i(\theta_{-i})}^{\theta_i} 1\,ds$.

→ So the transfers are $t_i(\theta_i,\theta_{-i}) = -\theta_i + (\theta_i - Y_i(\theta_{-i})) = -Y_i(\theta_{-i})$ if $\theta_i > Y_i(\theta_{-i})$, and 0 if $\theta_i \leq Y_i(\theta_{-i})$. We can think of θ_i^r, the value of θ_i s.t. $\varphi_i(\theta_i^r) = 0$, as a <u>reserve price</u>.

Special Case: Symmetric bidders, $F = F_1 = F_2 = \cdots = F_n$. Then, $\varphi_i(\theta_i) = \varphi(\theta_i)$ $= \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$. $\Rightarrow \varphi_i(\theta_i) > \varphi_j(\theta_j) \Leftrightarrow \varphi(\theta_i) > \varphi(\theta_j) \Leftrightarrow \theta_i > \theta_j$, and the reserve price $\varphi(\theta^r) = \emptyset$ is the same:

1) good goes to the highest bidder (highest valuation)
2) as long as it exceeds reserve price θ^r.
3) winner pays largest of the second highest bid and reserve price.

When $n=1$: posted price is optimal!